Nonlinear Evolution of Initially Elliptical Vortex in the Upper Layer of Two-layer Round Ocean

A.[illegible] Shavlyugin, Head of Research Department, Vladivostok State University of Economy, 41 Gogolya Str., 690014, Vladivostok, Russia

Abstract:The nonlinear evolution of initially elliptical vortex is studied in the framework of two-laye quasi-geostrophic contour dynamics (CD) model of a round ocean. The vortex is located in the upper layer an the lower layer is considered as passive. There were found several regimes of evolution depending on th initial ellipse aspect ratio. The critical values of aspect ratio that divide different modes of evolution have bee found numerically in a wide range of values of the model parameters.

Index Terms:Contour dynamics, Two-layer quasi-geostrophic model, Vortex patch, Kirchhoff vortex.

INTRODUCTION

FROM the theory of plane flows of an ideal incompressible fluid is known [1] that uniform elliptical vortex (Kirchhoff vortex) in an unbounded fluid rotates without changing its shape with a constant angular velocity $\Omega = \frac{ab}{(a+b)^2}\omega$, where a, b – axes of the ellipse, ω – vorticity inside it. Further studies [2] showed that the Kirchhoff vortex is stable to infinitesimal perturbations of its shape if $\frac{a}{b} \le 3$. Numerical studies of Kirchhoff vortex instability showed that there are two different kinds of evolution. For moderate values of aspect ratio thin filaments of vortex fluid are formed at the ends of major axis of the vortex. If aspect ratio becomes greater than some critical value, the initial ellipse is divided into two equal parts connected by a filament. For a highly elongated vortices the number of secondary parts may be greater than two.

It should be noted that in most of CD-based works concerning Kirchhoff vortex authors have considered flows in horizontally unbounded domain. Obviously, the real conditions in the ocean (sea) or laboratory experiments include rigid boundaries which should influence the observed phenomena. The first attempt to study vortex flows in a closed area by CD was made in [3] where barotropic ocean model for a circular domain was developed. Thereafter [4] in the framework of this model the study of nonlinear behavior of Kirchhoff vortex was carried out and some phenomena unknown in the case of unbounded fluid were found.

From the oceanography applications point of view the most interesting CD-based ocean model is the baroclinic one taking into account effects of vertical density stratification. Two-laye quasi-geostrophic CD-based model for horizontall unbounded ocean was developed first in [5] anc thereafter studies of two-layer vortex dynamic including the problems of two-layer axisymmetric vortex instability, vortex interactions, upper anc two-layer vortex merger and V-states were carriec out by many researchers (more information one car find in review [6]).

The main objectives of this paper are the study of stability properties of Kirchhoff vortex localized in the upper layer of two-layer round ocean and classification of different modes of instability in the most interesting nonlinear stage of the process.

GOVERNING EQUATIONS OF THE MODEL

The system under consideration is composed of two layers of density ρ and ρ + Δρ (Δρ « ρ) and thickness H1 and H2 (H1 « L*, H2 « L*, where L* being the linear horizontal scale of the system) for the upper and lower layer respectively.

In quasi-geostrophic approximation and non-dimensional form the potential vorticity (PV) Π conservation laws in layers can be written in form [7]

$$\frac{d_i \Pi_i}{dt} = 0,\ i = 1,2,... \qquad (1)$$

where indexes 1 and 2 refer to upper and lower layer respectively, the total derivative is

$$\frac{d_i}{dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x} + v_i \frac{\partial}{\partial y},$$

ui , vi – geostrophic velocities.

Let's make the assumption that Π is constant and nonzero at the initial time in the region S with the boundary C located in the upper layer. In accordance with (1) this property will be true at all subsequent times. Under this assumption expressions for pressure at point (x, y) can be written in form

$$p_1(x,y,t)=\Pi\iint_S G_L(x,y;\xi,\eta)d\xi d\eta-(1-d)p, \quad (2)$$

$$p_2(x,y,t)=dp, \quad (3)$$

$$p(x,y,t)=-\Pi\iint_S \widetilde{G}(x,y;\xi,\eta)d\xi d\eta, \quad (4)$$

where $\widetilde{G}=G_H-G_L$ (GH and GL are the Green's functions for the Helmholtz and Laplace equations respectively), d – relative thickness of the upper layer. Expressions (2-4) allow us to find pressure at any point of the flow and therefore we can calculate the components of geostrophic velocity using well-known relations

$$u_i=-\frac{\partial p_i}{\partial y},\quad v_i=\frac{\partial p_i}{\partial x}. \quad (5)$$

As known the Green's function for the Laplace's equation inside a circle of radius a has the form

$$G_L=\frac{1}{2\pi}\left(\log R-\log\frac{R^*r_0}{a}\right), \quad (6)$$

where

$$R=\left[(\xi-x)^2+(\eta-y)^2\right]^{1/2},$$

$$R^*=\left[(\xi^*-x)^2+(\eta^*-y)^2\right]^{1/2},$$

$$r_0=(\xi^2+\eta^2)^{1/2},\quad (\xi^*,\eta^*)=\frac{a^2}{r_0^2}(\xi,\eta).$$

Let's introduce the function

$$\Phi(x,y;\xi,\eta)=\int_0^1 G_L(x,y;x+(\xi-x)z,y+(\eta-y)z)zdz. \quad (7)$$

Using the Stokes's theorem and the identity

$$G_L=\left[(\xi-x)\Phi\right]_\xi+\left[(\eta-y)\Phi\right]_\eta,$$

we can rewrite the corresponding terms in (2-4) in form

$$\iint_S G_L d\xi d\eta=\oint_C \Phi\left[(\xi-x)d\eta-(\eta-y)d\xi\right]. \quad (8)$$

For the results of calculation (8) and other details of barotropic CD model for a circular domain see [4].

As shown in [8] the Green's function for the Helmholtz equation inside a circle of radius a can be written using polar coordinates in the form

$$G_H(r,\varphi)=\frac{1}{\pi}\left(\sum_{n=0}^{\infty}\frac{K_n(ka)}{\mu_n I_n(ka)}I_n(kr)I_n(kr_0)\cos n(\theta-\varphi)-\frac{1}{2}K_0(kR)\right) \quad (9)$$

where Kn and In are the modified Bessel functions of order n, $r=(x^2+y^2)^{1/2}$, k – parameter characterizing the baroclinic effects. In the case of last term of (8) one can use the symmetry of argument of Bessel function with respect to (x, y) and (ξ, η) to transform integrals over S in (5) to integrals over C. To do this for the case of first term of (8) let's introduce the function

$$Q_n(r_0)=\int_0^{r_0} zI_n(kz)dz,$$

and then using the Stokes's theorem we can write

$$\iint_S\sum_{n=0}^{\infty}\frac{K_n(ka)}{\tau_n I_n(ka)}I_n(kr)I_n(kr_0)\cos n(\theta-\varphi)r_0dr_0d\theta=$$

$$=\oint_C\frac{K_n(ka)}{\tau_n I_n(ka)}I_n(kr)Q_n(r_0)\cos n(\theta-\varphi)d\theta \quad (10)$$

Relations (2-10) allow us to calculate velocity components at any point of domain and hence determine time evolution of vortex patches by solving the system of ordinary differential equations of motion of fluid particles lying on C

$$\frac{dx}{dt}=u,\quad \frac{dy}{dt}=v. \quad (11)$$

NUMERICAL EXPERIMENTS

Consider the special case when S is an ellipse with semiaxis a and b. We will classify the modes of evolution depending on such parameters as the ratio of semiaxis $\varepsilon=\frac{a}{b}$, the square of the ellipse $S=\pi ab$, the relative thickness of the upper layer d, and the baroclinic parameter k.

Experiments have shown that there are two different modes of evolution. If ε is less than approximately 5.0, vortex performs a quasi-periodic oscillations about some equilibrium shape and after a few revolutions loses its symmetry and is divided into two unequal parts (sometimes thin filament forms instead of one of these parts). The time required for the loss of symmetry of the system depends on the value of semiaxis ratio bat this phenomenon was observed in all cases. Fig. 1 shows the example of this mode of evolution for the set values of parameters: d=0.5, k=5.0, ε=4.0, S=0.503.

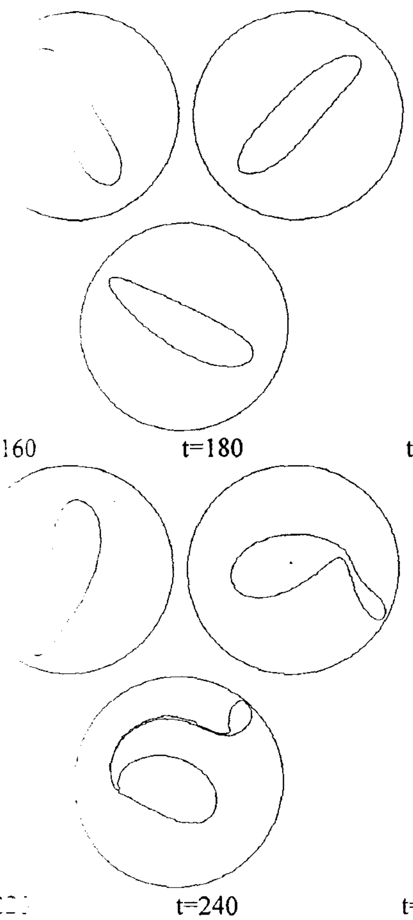


Fig. 1 The mode of evolution of elliptical vortex with the loss of symmetry of the system.

For fixed values of S, k, d there is a critical value of ε above which another mode of evolution takes place. In this case vortex is divided to two equal parts connected by thin filament during first revolution. Later formed parts show a tendency to periodic merge/split. The example of this behavior is shown in Fig. 2.

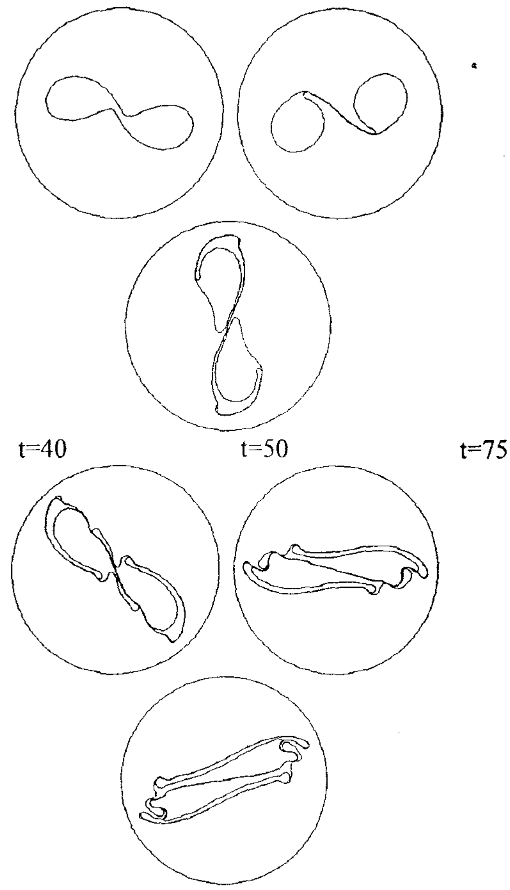
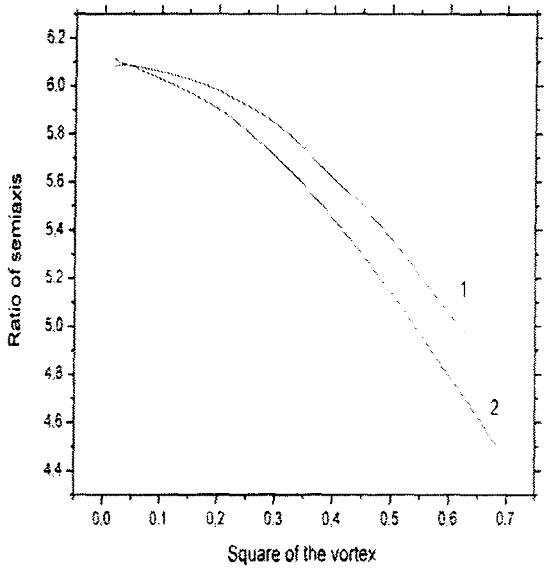


Fig. 2 The mode of evolution of elliptical vortex wi periodic merge/split.

The critical value of ε depends on all oth parameters of the problem. Fig. 3 illustrates t influence of d and S on ε. Lines 1 and 2 were drav for k=1.0, d=0.5 and k=1.0, d=0.2 respectivel Values of S and ε above each curve correspond to merge/split mode of evolution. It can concluded that the decrease in the thickness of t upper layer leads to the expansion of the paramete field in which the mode of merge/split is observed.



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