On Unsteady Heat Effect in Center of the Elastic-Plastic Disk

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***Abstract*—The present study is devoted to the boundary value problems of the thermoelastoplasticity with temperature dependent yield stress. The one dimensional problem of the residual stresses formation in the thin elastic-plastic disk under a given thermal action is analytically solved. The analytical solution is found for the plastic flow and unloading process in center of the solid disk under unsteady heat effect. For calculation we applied the maximum reduced stress (Ishlinsky–Ivlev‘s) yield condition with temperature dependent yield stress. It is shown that secondary plastic flow may arise in the unloading processes, which significantly redistributes the final residual stresses. The fields of residual displacements and stresses are computed and graphically analyzed.**

***Index Terms*—elasticity, heat conduction, Ishlinsky–Ivlev‘s yield condition, maximum reduced stress, plasticity, residual strain, thermal stress.**

I. Introductory

As it’s well known, the high temperature gradients are able to lead to irreversible deformation and residual stresses formation in a material [1], [2], [3]. Fast heating increases the temperature difference that it’s also leads to increase of the shearing stress. The state of material unloading or shearing stress decreasing is always occurs to the extent of the temperature equalization. In such cases, unsteady temperature field forms the time-dependent domains of the irreversible strains. The plastic strains accumulates in these domains. The difficulty of a calculation of the irreversible deformations under an non-stationary thermal field deals with determination of the elastic-plastic border position. In the general case, the position of the elastic plastic borders could depend on the accumulated irreversible deformation level. The Tresca‘s [4], [5] or (Huber) von Mises‘s [6], [7] yield conditions are usually used for calculation of plastic flow processes [1], [2], [3], [9], [8], [10], [11]. However there are problems in which Tresca‘s yield condition using leads to the incorrect results. One of these problems is presented in the paper. We consider the process of a plastic strain forming due to fast heating of the central part of the elastic-plastic disk. The peculiarity of this problem is an assumption of Ishlinsky–Ivlev‘s (maximum reduced stress) yield condition [5], [14] with temperature dependent yield stress [15] as an yield surface.

The theory of thermal plasticity [16] including the thermal stress theory in the flow conditions allowed us to obtain some useful answers to a few questions of technological practice. There are clear perspectives for its development in this direction. This publication is intended to substantiate this.

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II. Governing Equations Reminder

We assume that, till the time $t = 0$, the disk is in free state at room temperature T_0. We also assume that the isotropic elastic-plastic material of the disk obeys the PrandtlReiss-type model [5], [16]. The infinitesimal strains d_{ij} are composed of elastic (reversible) e_{ij} and plastic (irreversible) p_{ij} residual strains. Thus, in cylindrically symmetric case we can obtain

$$\begin{aligned} d_{rr} &= u_{r,r} = e_{rr} + p_{rr}, \\ d_{\varphi\varphi} &= \frac{u_r}{r} = e_{\varphi\varphi} + p_{\varphi\varphi}, \\ d_{\varphi\varphi,r} &+ \frac{d_{\varphi\varphi} - d_{rr}}{r} = 0. \end{aligned} \tag{1}$$

u_r is a radial component of the displacement vector. The index after comma denotes partial derivative with respect to the corresponding spatial coordinate.

The level and distribution of elastic strains and the temperature over the plate determine the stresses in the disk which obey the Duhamel-Neumann law

$$\begin{aligned} \sigma_{rr} &= \frac{4\mu(\lambda+\mu)e_{rr}}{(\lambda+2\mu)} + \frac{2\lambda\mu e_{\varphi\varphi}}{(\lambda+2\mu)} - \frac{2\alpha(3\lambda+2\mu)T}{(\lambda+2\mu)}, \\ \sigma_{\varphi\varphi} &= \frac{4\mu(\lambda+\mu)e_{\varphi\varphi}}{(\lambda+2\mu)} + \frac{2\lambda\mu e_{rr}}{(\lambda+2\mu)} - \frac{2\alpha(3\lambda+2\mu)T}{(\lambda+2\mu)}, \\ e_{zz} &= \alpha T - \frac{\lambda(e_{rr}+e_{\varphi\varphi})}{(\lambda+2\mu)}, \end{aligned} \tag{2}$$

where T is the difference between current and initial temperature of the disk; λ, μ are the Lame’s parameters; α is the coefficient of linear thermal expansion.

The thermal stresses inside the disk should satisfy to equilibrium equation

$$\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \tag{3}$$

The Ishlinsky–Ivlev‘s yield condition [5] is used as yield criteria:

$$\max\{|\sigma_1 - \sigma|, |\sigma_2 - \sigma|, |\sigma_3 - \sigma|\} = \frac{4k(T)}{3}. \tag{4}$$

Herein, σ_i is the principal components of the Cauchy‘s stress tensor, $\sigma = (\sigma_1 + \sigma_2 + \sigma_3)/3$. We assume that the yield stress is a linear function on the temperature $k(T) = k0(1 - \beta T)$. The local thermal effect in plane stress states leads to satisfying of the yield condition (4) in the form of

$$\sigma_{rr} + \sigma_{\varphi\varphi} = -4k(T). \tag{5}$$

The plastic incompressibility condition follows from the plastic flow rule associated with equation (5) and suggests

$$p_{rr} + p_{\varphi\varphi} + p_{zz} = 0, \quad p_{rr} = p_{\varphi\varphi}. \tag{6}$$

In case, when the yield condition (4) doesn't valid (for example in an unloading state), it is conveniently to determine the stress-strain state from equations (2-3) as following differential equation:

$$\gamma \left(P_{r,r} + \alpha T_{,r}\right) + 2\sigma_{rr,r} + (r\sigma_{rr,r})_{,r} = 0, \tag{7}$$

where $P_r(r)$ is the accumulated irreversible strain $p_{rr}(r,t)$ at the plastic flow state, $\gamma = \mu(3\lambda + 2\mu)/(\lambda + \mu)$. If we integrate equation (7), one can derive:

$$\begin{aligned} \sigma_{rr} &= -\frac{\gamma}{r^2}\left(\int_l^r \rho P_r(\rho)\, d\rho + \alpha \int_l^r \rho T(\rho,t)\, d\rho\right) + \\ &\quad + A(t) + \frac{B(t)}{r^2}, \\ \sigma_{\varphi\varphi} &= A(t) - \frac{B(t)}{r^2} - \gamma P_r(r) - \alpha\gamma T(r,t) + \\ &\quad + \frac{\gamma}{r^2}\left(\int_l^r \rho P_r(\rho)\, d\rho + \alpha \int_l^r \rho T(\rho,t)\, d\rho\right), \end{aligned} \tag{8}$$

Herein, l is the coordinate in the domain, where equation (7) are true; $A(t), B(t)$ are unknown functions.

Therefore, for the radial displacement from equations (8) one can computed

$$\begin{aligned} u_r &= \frac{\gamma}{2\mu r}\left(\int_l^r \rho P_r(\rho)\, d\rho + \alpha \int_l^r \rho T(\rho,t)\, d\rho\right) + \\ &\quad + \frac{r}{2\omega} A(t) - \frac{1}{2r\mu} B(t). \end{aligned} \tag{9}$$

where $\omega = \mu(3\lambda + 2\mu)/(\lambda + 2\mu)$.

Integrating equilibrium equation (3) on the condition (5) for stresses in plastic flow domains one reads:

$$\begin{aligned} \sigma_{rr} &= -\frac{4}{r^2}\int_l^r \rho k(\rho,t)\, d\rho + \frac{1}{r^2} X(t), \\ \sigma_{\varphi\varphi} &= \frac{4}{r^2}\int_l^r \rho k(\rho,t)\, d\rho - \frac{1}{r^2} X(t) - 4k(r,t), \end{aligned} \tag{10}$$

where $X(t)$ is an unknown function.

It is necessary to calculate the components of the plastic strains for finding of the radial displacement in the plastic flow domain. Let's use equation (1), (6) by substituting in equation (2). After rearrangements we find:

$$p_{rr} = p_{\varphi\varphi} = \frac{4}{\gamma} k(r,t) - \alpha T(r,t) + Y(t) \tag{11}$$

The formula for the radial displacement with the consideration of the calculated strains (11) rewrite as follows

$$u_r = \frac{2}{\mu r}\int_l^r \rho k(\rho,t)\, d\rho - \frac{1}{2\mu r} X(t) + rY(t). \tag{12}$$

Here $Y(t)$ is an unknown function.

III. Problem Statement and Numerical Results

The material parameters corresponding to cooper were used for further computations as shown in Table I.

We assume that, in the time $t = t_0$ the boundary of the circle area $0 < r < R_0$ is stresses free

$$\sigma_{rr}(R) = 0 \tag{13}$$

TABLE I
Material constants

Symbol	Quantity	Value
R	Radius of disk	0.2 m
R_0	Radius of heat influence area	0.02 m
T_k	Maximum temperature difference	700 K
β	Parameter of decreasing yield stress	0.4
k_0	Initial yield stress at $T = 0$	80×10^6 Pa
x	Rate of heating	0.5
λ	Lamé constant	91.2×10^9 Pa
μ	Lamé constant (shear modulus)	42.9×10^9 Pa
α	coefficient of linear thermal expansion	17×10^{-6} K^{-1}

and heated by law

$$T = T_k(1 - e^{-xt}), \tag{14}$$

where x is the positive heat rate.

Solution of the heat conduction equation (see in details [17]) with the boundary conditions (14) and

$$T_{,r}(R,t) = 0$$

and initial condition

$$T(r,0) = 0$$

corresponds to fast heating of a material up to the predetermined temperature T_k as shown on Fig. 1.

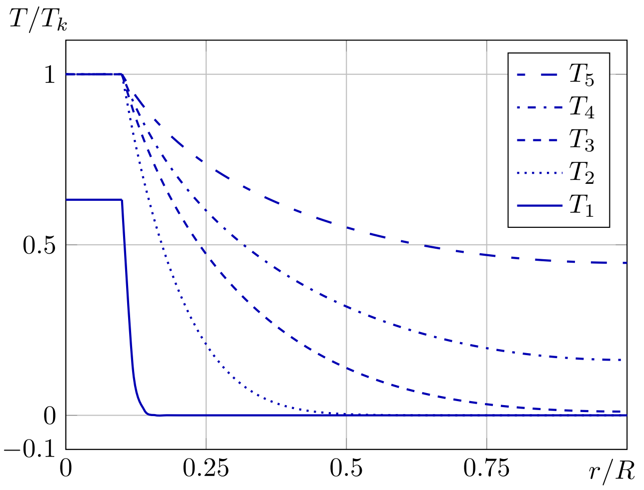

Fig. 1. Temperature field. $T_1 = T(t_1)$, $T_2 = T(t_2)$, $T_3 = T(t_3)$, $T_4 = T(t_4)$, $T_5 = T(t_5)$, $t_0 < t_1 < t_2 < t_3 < t_4 < t_5$.

Till the time $t = t_p$ the strain-stress state is determining by equations (8)–(9), where $a = l$, $P_r = 0$ and the unknown functions $A(t), B(t)$ are computed by

$$A = \frac{\alpha\gamma}{R^2}\int_0^R \rho T(\rho,t)\, d\rho, \quad B = 0. \tag{15}$$

From the time $t = t_p$ a yield condition (5) is valid in plastic flow domain $0 < r < a$ with elastic-plastic border $a(t)$ for $t > t_p$.

Let's remark here, that Tresca's yield condition in the considering case is transformed to

$$\sigma_{rr} = \sigma_{\varphi\varphi} = -2k.$$

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Substituting this condition into the equilibrium equation (3) we could get contradiction $k_{,r}(r,t)=0$ in the elastic domain $R_0<r<a$.

The strain-stress state in the plastic flow domain $0<r<a$ is determined by equations (10)–(12), where $l=0$. The strain-stress state in the unloading domain $a<r<R$ can be calculated by equations (8)–(9), where $l=a$, $P_r=0$. Equations (8)–(12) contain new time dependent functions $A(t), B(t), X(t), Y(t)$. They are possible to find from the constraints for continuity of the stresses and radial displacement across elastic-plastic border a by formulae

$$A=\frac{4\int\limits_0^a \rho k(\rho,t)\,d\rho+\alpha\gamma\int\limits_a^R \rho T(\rho,t)\,d\rho}{R^2-a^2},$$
$$B=\frac{4R^2\int\limits_0^a \rho k(\rho,t)\,d\rho+\alpha\gamma a^2\int\limits_a^R \rho T(\rho,t)\,d\rho}{a^2-R^2}, \qquad (16)$$
$$X=0,\ Y=\frac{8\int\limits_0^a \rho k(\rho,t)\,d\rho+\alpha\gamma\int\limits_a^R \rho T(\rho,t)\,d\rho}{\gamma(R^2-a^2)}.$$

The value a is possible to find numerically from the equation $p_{rr}(a,t)=0$ for the different times $t>t_p$.

Thermal stresses during plastic flow are presented on the Fig. 2.

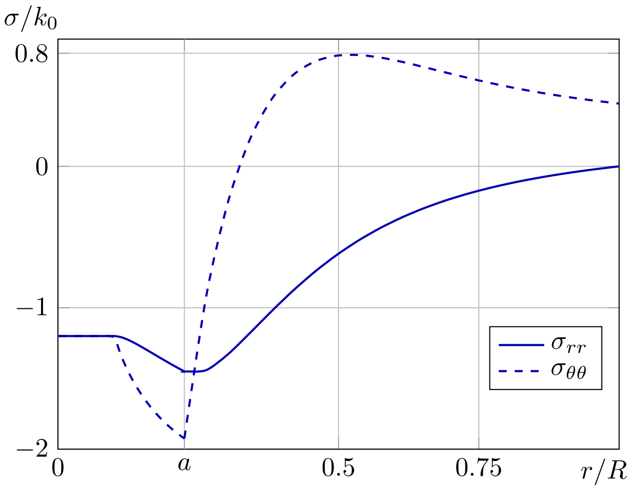

Fig. 2. Thermal Stresses. Plastic Flow.

It's possible to determine the origin time of a material unloading by means of two equivalent ways. On the one hand, the material unloading leads to stop of the plastic strains growth. So, the condition of material unloading could be written in the form $p_{rr,t}(b,t)=0$, where b is the second elastic-plastic border. On the other hand, if we assume, that the material unloading was appearing in the time $t=t_u$, it could be yield condition failure (5). In this case, the condition of unloading could be proposed in the form $\sigma_{rr,t}(b,t)+\sigma_{\varphi\varphi,t}(b,t)=-4k_{,t}(b,t)$. Thus, there are 3 different domains, when $t>t_u$:

- the unloading domain ($0<r<b$), $l=0$ with accumulated irreversible strains P_r,
- the plastic flow domain ($b<r<a$), $l=b$,
- the unloading domain ($a<r<R$), $l=a$ without irreversible strains.

The equations (8)–(12) for stresses and radial displacement in each of these domains contain own unknown functions $A(t), B(t), X(t), Y(t)$, which are possible to find from for the stress-strain state parameters continuity conditions on an elastic-plastic borders a, b.

In the unloading domain $a<r<R$:

$$A(t)=M_1\int\limits_b^a \rho k(\rho,t)d\rho+\mu M_2\int\limits_0^b \rho P_r(\rho)d\rho+$$
$$+\alpha\mu M_2\int\limits_0^b \rho T(\rho,t)d\rho-\frac{1}{4}\left(\alpha\gamma M_2\right)\int\limits_a^b \rho T(\rho,t)d\rho+$$
$$+\alpha\mu M_2\int\limits_a^R \rho T(\rho,t)d\rho,$$

$$B(t)=-R^2M_1\int\limits_b^a \rho k(\rho,t)\,d\rho-R^2\mu M_2\int\limits_0^b \rho P_r(\rho)\,d\rho-$$
$$-R^2\alpha\mu M_2\int\limits_0^b \rho T(\rho,t)\,d\rho+\frac{R^2M_1\alpha\gamma\int\limits_a^b \rho T(\rho,t)d\rho}{4(\lambda+2\mu)}+$$
$$+\left(b^2-a^2\right)\alpha\gamma M_1\int\limits_a^R \rho T(\rho,t)d\rho,$$

$$M_1=\frac{4(\lambda+2\mu)}{4R^2(\lambda+\mu)-a^2(\lambda+2\mu)+b^2(\lambda+2\mu)},$$

$$M_2=\frac{4(3\lambda+2\mu)}{4R^2(\lambda+\mu)-a^2(\lambda+2\mu)+b^2(\lambda+2\mu)}.$$

In the unloading domain $0<r<b$:

$$A(t)=\frac{\left(R^2-a^2\right)\alpha\gamma M_2\int\limits_a^b \rho T(\rho,t)\,d\rho}{4b^2}+$$
$$+\frac{\mu M_2S_a\int\limits_0^b \rho P_r(\rho)\,d\rho}{4b^2(\lambda+\mu)}+\frac{S_RM_1\int\limits_b^a \rho k(\rho,t)\,d\rho}{b^2(\lambda+2\mu)}+$$
$$+\frac{\alpha\mu M_2S_a\int\limits_0^b \rho T(\rho,t)\,d\rho}{4b^2M(\lambda+\mu)}+\frac{\alpha\mu M_2S_a\int\limits_a^R \rho T(\rho,t)\,d\rho}{4b^2(\lambda+\mu)},$$

$$B(t)=0,$$

$$S_R=b^2(\lambda+2\mu)+R^2(3\lambda+2\mu),$$

$$S_a=b^2(\lambda+2\mu)+a^2(3\lambda+2\mu).$$

In the plastic flow domain $b<r<a$:

$$X(t)=-\left(R^2-a^2\right)\mu M_2\int\limits_0^b \rho P_r(\rho)d\rho-$$
$$-\left(R^2-a^2\right)\alpha\mu M_2\int\limits_0^b \rho T(\rho,t)d\rho+$$
$$+\frac{1}{4}\left(\left(R^2-a^2\right)\alpha\gamma M_2\right)\int\limits_a^b \rho T(\rho,t)d\rho+$$
$$+\frac{\left(\alpha\gamma M_1S_a\right)\int\limits_a^R \rho T(\rho,t)d\rho}{4(\lambda+2\mu)}+\frac{\left(M_1S_R\right)\int\limits_b^a \rho k(\rho,t)d\rho}{\lambda+2\mu},$$

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$$Y(t) = \frac{2M_1 \int\limits_b^a \rho k(\rho,t)d\rho}{\gamma} + \frac{(2\mu M_2)\int\limits_0^b \rho P_r(\rho)d\rho}{\gamma} + \frac{(2\alpha\mu M_2)\int\limits_0^b \rho T(\rho,t)d\rho}{\gamma} - \frac{1}{2}(\alpha M_2)\int\limits_a^b \rho T(\rho,t)d\rho + \frac{(2\alpha\mu M_2)\int\limits_a^R \rho T(\rho,t)d\rho}{\gamma}.$$

The elastic-plastic borders and a level of the irreversible strains were computed numerically.

It's needed to solve the system of nonlinear equations relative to unknown values a, b, P_r in the times t_i:

$$\begin{cases} p_{rr}(b,t_i) = P_r(b), \\ p_{rr,t}(b,t_i) = 0, \\ p_{rr}(a,t_i) = 0. \end{cases} \qquad (17)$$

Integrals, containing the irreversible strains in this system of equations, are changing by approximation according to the method of trapezium [3].

The condition $a = b$ at the time t_n means the full unloading of the disk. Material of the disk is deformed elastically for $t > t_n$.

The results of calculation of the residual stresses and radial displacement are presented on the Fig. 3 and Fig. 4 respectively.

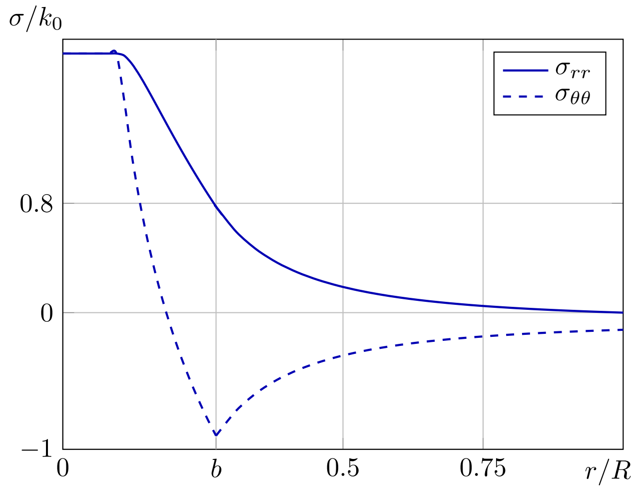

Fig. 3. Thermal Stresses. Unloading.

IV. Conclusion

The results of the calculations suggest that irreversible strains and stresses are decreasing the disk size under cooling down (Fig. 4). The high level of the positive residual stresses in the center of the disk (Fig. 3) means the possibility of the secondary plastic flow occurrence with the opposite sign in the yield condition (5). Moreover, the size of the domain of the thermal influence impacts on the final distribution of the residual stresses. The process of the plastic flow could be appeared in the distant from the disk center with increasing of the radius [12]. The next study of the similar problems

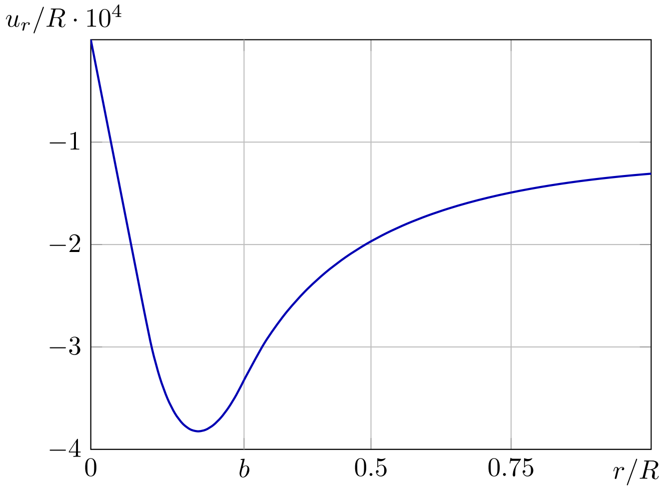

Fig. 4. Radial Displacement.

could be dealing with simultaneous taking account of the relation of the yield stress on temperature and linear strain hardening [13] under the yield condition (5).

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