ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2012, Vol. 6, No. 4, pp. 431–442. © Pleiades Publishing, Ltd., 2012.
Original Russian Text © A.A. Burenin, L.V. Kovtanyuk, A.S. Ustinova, 2012, published in Sibirskii Zhurnal Industrial'noi Matematiki, 2012, Vol. XV, No. 2, pp. 43–55.

Viscosimetric Flow of an Incompressible Elastoviscoplastic Material under the Presence of a Lubricant on the Boundary Surfaces

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Received July 18, 2011

Abstract—Generalizing the model of large deformations by accounting for the viscous properties of materials, we obtain the analytical solutions of some quasistatic boundary value problems concerning the viscosimetric flows of an elastoviscoplastic material in the gap between the rigid coaxial cylindrical surfaces when, in the neighborhood of one of the rigid cylinders (either internal or external), there is a layer of an elastic non-Newtonian lubricant, and the rigid adhesion conditions are satisfied on boundary surfaces. The conditions are studied of origination of a flow in the lubricant layer and in the basic material. The values of the maximum velocity are specified under which the flow does not tresspass the lubricant layer.

DOI: 10.1134/S1990478912040047

Keywords: *elasticity, viscoplasticity, large deformations, residual stresses*

INTRODUCTION

In lubrication theory, the elasticity of a lubricating liquid is usually neglected since its shear modulus is assumed to be negligibly small in comparison with that of the rubbing parts. Therefore, lubrication is modeled by a viscous or viscoplastic liquid whereas the rubbing parts are assumed, most often, to be rigid. Let us note additionally that the sophistication in the model connected with the presence of the fluidity limit turned out important not only because the majority of lubricating materials manifest such properties but also because the very character of rapid movements in the thin layers presupposes the formation of stagnant zones near the asperities of the boundary surfaces. It leads to the fact that the viscous resistance to movement for viscoplastic lubrication turns out smaller than in a similar viscous case. The wear-and-tear and the fatigue strength of the materials joined by the lubricant layer are connected in that case only with the direct contact of the rubbing parts due to extrusion of the lubricant layer; i.e., essentially, due to the defects in lubrication.

If the elastic properties of a lubricant material were taken into account then there would be present some different effect in decreasing the long-term durability connected with the transfer of deformations through the lubricant layer. Then the developing deformations give rise to a field of stresses in the contacting bodies, which under certain conditions results in their irreversible deformations and formation of residual stresses. These processes significantly influence the fatigue strength of the moving constructional elements, and it is exactly the influence of these processes that is supposed to diminish the lubricant layer.

Accounting for the elastic properties in the lubricant layer and the elastoplastic properties of the materials contacting through lubrication of the bodies encounters considerable mathematical difficulties. The deformations in the layer cannot be considered small; hence, it is necessary to engage the theory of large deformations of the bodies with elastic, plastic, and viscous properties [1–11]. Under certain

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conditions, the same turns out necessary while modeling the deformation process of the contacting bodies.

Let us use the model that we proposed earlier [10], where the two hypotheses were introduced: during unloading the components of the tensor of irreversible deformations change in the same way as during rigid movement; the reversible deformations completely determine the stress field in a medium. It turns out [10, 11] that these conditions, natural for classical elastoplasticity [12], suffice to construct a mathematical model in whose framework it is possible to formulate and solve the boundary value problems of the theory [11, 13–15] including the viscosimetric flow problems [16].

1. THE BASIC MODEL RELATIONS

In the rectangular Cartesian system of the Euler spatial coordinates, the kinematics of a medium is determined by the equations [11]

$$
\begin{gathered}
d_{ij} = e_{ij} + p_{ij} - \frac{1}{2}\, e_{ik}e_{kj} - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{ks}e_{sj}, \\
\frac{De_{ij}}{Dt} = \varepsilon_{ij} - \varepsilon^p_{ij} - \frac{1}{2}\big((\varepsilon_{ik} - \varepsilon^p_{ik} + z_{ik})e_{kj} + e_{ik}(\varepsilon_{kj} - \varepsilon^p_{kj} - z_{kj})\big), \\
\frac{Dp_{ij}}{Dt} = \varepsilon^p_{ij} - p_{ik}\varepsilon^p_{kj} - \varepsilon^p_{ik}p_{kj}, \qquad \frac{Dn_{ij}}{Dt} = \frac{dn_{ij}}{dt} - r_{ik}n_{kj} + n_{ik}r_{kj}, \\
\varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \qquad v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_{i,j}v_j, \qquad u_{i,j} = \frac{\partial u_i}{\partial x_j}, \\
r_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) + z_{ij}(\varepsilon_{sk}, e_{sk}).
\end{gathered}
\tag{1.1}
$$

In (1.1), d_{ij} are the components of the Almansi deformation tensor; e_{ij} and p_{ij} are their reversible and irreversible components; u_i and v_i are the components of the displacement vectors and the velocities of the medium points; $\frac{D}{Dt}$ is the objective derivative of the tensors with respect to time shown for an arbitrary tensor n_{ij}; and ε^p_{ij} (the source in the transfer equation for the tensor of irreversible deformations) are the components of the tensor of the plastic deformation rates. The presence of a nonlinear component z_{ij} of the rotation tensor r_{ij}, which is completely written in [10], is connected with the requirement of constancy of the plastic deformations tensor p_{ij} in unloading processes.

The stresses in the medium are entirely determined by the reversible deformations and, for an isotropic incompressible material, are related to them via the equations

$$
\begin{gathered}
\sigma_{ij} = -p\delta_{ij} + \frac{\partial W}{\partial d_{ik}}(\delta_{kj} - 2d_{kj}) \qquad \text{for} \quad p_{ij} \equiv 0, \\
\sigma_{ij} = -p_1\delta_{ij} + \frac{\partial W}{\partial e_{ik}}(\delta_{kj} - e_{kj}) \qquad \text{for} \quad p_{ij} \neq 0, \\
W = -2\mu J_1 - \mu J_2 + bJ_1^2 + (b-\mu)J_1J_2 - \chi J_1^3 + \dots, \\
J_k = \begin{cases} L_k, & p_{ij} \equiv 0 \\ I_k, & p_{ij} \neq 0, \end{cases} \qquad L_1 = d_{kk}, \qquad L_2 = d_{ik}d_{ki}, \\
I_1 = e_{kk} - \frac{1}{2}e_{sk}e_{ks}, \qquad I_2 = e_{st}e_{ts} - e_{sk}e_{kt}e_{ts} + \frac{1}{4}e_{sk}e_{kt}e_{tn}e_{ns}.
\end{gathered}
\tag{1.2}
$$

In (1.2), σ_{ij} are the components of the Euler–Cauchy stress tensor, p and p_1 are the additional hydrostatic pressures, W is the elastic potential, μ is the shear modulus, and b and χ are some constant parameters of the material. The Tresk load function [17] is used as the plastic potential:

$$
\max|\sigma_i - \sigma_j| = 2k + 2\eta \max\left|\varepsilon^p_k\right|, \tag{1.3}
$$

where k is the fluidity limit, η is the viscosity coefficient, while σ_i and ε^p_k are the principal values of the tensors of stresses and the rates of plastic deformations.

The connection between the rates of the irreversible deformations and the stresses is established by the associated law of plastic flow:

$$\varepsilon_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}}, \qquad f\big(\sigma_{ij}, \varepsilon_{ij}^p\big) = k, \qquad \lambda > 0. \tag{1.4}$$

2. REVERSIBLE DEFORMATION AND VISCOPLASTIC FLOW

Let us consider the case when the lubricant layer is situated next to the internal rigid cylinder which is rotating around its axis. The mechanical properties of the basic material $r_1 \le r \le R$ are given by the parameters μ_1, b_1, χ_1, k_1, and η_1; the analogous mechanical parameters of the layer $r_0 \le r \le r_1$ are denoted by μ_2, b_2, χ_2, k_2, and η_2, while $k_2 < k_1$. We assume that the adhesion conditions hold on the boundary surfaces; and the boundary conditions on the external rigid cylinder are of the form

$$\bar{u} = \bar{v} = 0 \qquad \text{for} \quad r = R. \tag{2.1}$$

When the points of the medium move along the circles, the components of the displacement vector are given by the dependencies $u_r = r(1 - \cos\theta(r,t))$ and $u_\varphi = r\sin\theta(r,t)$, where $\theta(r,t)$ is the central angle of twisting. The nonzero components of the Almansi deformation tensor are as follows:

$$d_{rr} = -\frac{1}{2}\, g^2, \qquad d_{r\varphi} = \frac{1}{2}\, g, \qquad g = r\frac{\partial \theta}{\partial r}.$$

The equations (1.2) for the deformation components yield the following relations accurate to within the terms of the second order of smallness with respect to deformations:

$$\sigma_{rr} = \sigma_{zz} = -(p + 2\mu) - \frac{1}{2}\,(b+\mu)g^2 = -s, \qquad \sigma_{\varphi\varphi} = -s + \mu g^2, \qquad \sigma_{r\varphi} = \mu g.$$

With the selected arrangement of the layer, the plastic flow always originates near the internal rigid surface $r = r_0$. This happens when the stressed state comes out to the loading surface $\sigma_{r\varphi}|_{r=r_0} = -k_2$.

Integrating the equations of equilibrium (the quasistatic approximation)

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \qquad \frac{\partial \sigma_{r\varphi}}{\partial r} + 2\frac{\sigma_{r\varphi}}{r} = 0 \tag{2.2}$$

and using the condition of coincidence of displacements for $r = r_1$ and $r = R$, we obtain

$$\theta = A(c), \qquad A(c) = \frac{c}{2\mu_1}\big(1/R^2 - 1/r^2\big)$$

in the domain $r_1 \le r \le R$ and

$$\theta = A_1(c), \qquad A_1(c) = \frac{c}{2\mu_1}\big(1/R^2 - 1/r_1^2\big) + \frac{c}{2\mu_2}\big(1/r_1^2 - 1/r^2\big)$$

in the layer $r_0 \le r \le r_1$; here c is an unknown constant of integration.

Using the plasticity condition, we find that $c = -k_2 r_0^2$ at the start of the plastic flow. The rotation angle on which the rigid cylinder should be rotated to initiate the plastic flow can be obtained by the formula

$$\theta_0 = \frac{k_2}{2\mu_2}\big(1 - r_0^2/r_1^2\big) + \frac{k_2}{2\mu_1}\big(r_0^2/r_1^2 - r_0^2/R^2\big). \tag{2.3}$$

Starting from the time $t = t_0 = 0$, some domain of viscoplastic flow develops bounded by the surfaces $r = r_0$ and $r = x_1(t)$ $(r_0 \le r \le x_1(t))$.

According to (1.1), the following kinematic relations hold for the components of the velocity vector and the tensor of the deformation rates:

$$\begin{gathered}
v_\varphi = r\frac{\partial \theta}{\partial t} = r\omega, \qquad \varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\right) = \frac{\partial d_{r\varphi}}{\partial t} = \frac{1}{2}\, r\frac{\partial^2 \theta}{\partial r \partial t}, \\
\varepsilon_{r\varphi} = \varepsilon_{r\varphi}^e + \varepsilon_{r\varphi}^p = \frac{\partial e_{r\varphi}}{\partial t} + \frac{\partial p_{r\varphi}}{\partial t}, \qquad \varepsilon_{rr}^p = \frac{\partial p_{rr}}{\partial t} + 2p_{r\varphi}\big(r_{\varphi r} + \varepsilon_{r\varphi}^p\big), \\
\varepsilon_{\varphi\varphi}^p = \frac{\partial p_{\varphi\varphi}}{\partial t} + 2p_{r\varphi}\big(r_{r\varphi} + \varepsilon_{r\varphi}^p\big), \qquad \varepsilon_{rr}^p = -\varepsilon_{\varphi\varphi}^p = -2\varepsilon_{r\varphi}^p e_{r\varphi}.
\end{gathered} \tag{2.4}$$

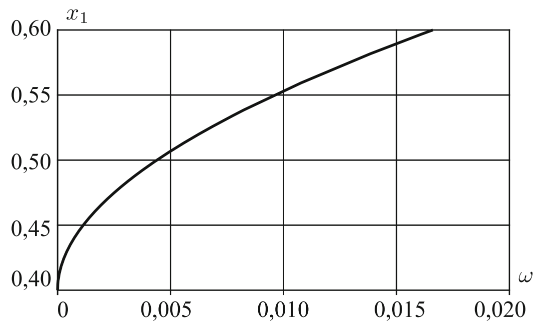

Fig. 1. The development of the region of viscoplastic flow

Let us calculate the parameters of the stressed-deformed state at some time $t = t_1 \geq t_0$. Neglecting the inertia forces in the domains of elastic deformation, we obtain

$$\sigma_{r\varphi} = c_1/r^2, \qquad \omega = 0, \qquad c_1 = c(t_1), \qquad \theta = \begin{cases} A(c_1), & r_1 \leq r \leq R, \\ A_1(c_1), & x_1(t) \leq r \leq r_1. \end{cases} \tag{2.5}$$

From the second equation of (1.2) we have for the stress components in the plastic flow domain

$$\sigma_{rr} = \sigma_{zz} = -(p_1 + 2\mu) - \frac{1}{2}(b + \mu)e_{r\varphi}^2 = -s_1(t),$$
$$\sigma_{\varphi\varphi} = -s_1(t) + 4\mu e_{r\varphi}^2, \qquad \sigma_{r\varphi} = 2\mu e_{r\varphi}.$$

At the same time, integrating the equilibrium equations, we have

$$\sigma_{r\varphi} = m(t_1)/r^2, \qquad e_{r\varphi} = m(t_1)/(2\mu r^2).$$

It follows from the conditions of continuity of the stress components that $m(t_1) = c_1$ and $s(t_1) = s_1(t_1)$. According to (1.3) and (1.4), we have

$$\sigma_{r\varphi} = -k + \eta\varepsilon_{r\varphi}^p, \qquad \lambda = -\varepsilon_{r\varphi}^p/\big(k - \eta\varepsilon_{r\varphi}^p\big).$$

Using (2.4) and the continuity condition for the velocities and displacements, we find in the region of viscoplastic flow that

$$\varepsilon_{r\varphi}^p = \frac{1}{\eta_2}(k_2 + c_1/r^2), \qquad p_{r\varphi} = \frac{t}{\eta_2}(k_2 + c_1/r^2), \qquad c_1 = -k_2x_1^2, \qquad \omega = F(c_1, r, x_1),$$
$$\theta = tF(c_1, r, x_1) + A_4(c_1), \qquad F(c_1, r, x_1) = \frac{2}{\eta_2}\left(k_2 \ln\frac{r}{x_1} + \frac{c_1}{2}\big(1/x_1^2 - 1/r^2\big)\right).$$

Specifying the rotation velocity of the internal cylinder, we obtain an equation for determining the elastoplastic boundary $x_1(t)$:

$$\omega(r_0, t_1) = F(c_1, r_0, x_1). \tag{2.6}$$

Under the increase of the rotation velocity of the rigid cylinder, the region of viscoplastic flow enlarges; and, at a certain moment, the boundary x_1 reaches the external surface of the layer $r = r_1$. In Fig. 1, the development of this region is shown in its dependence on the rotation velocity of the internal rigid cylinder.

The calculations were conducted under the following values of the parameters:

$$\begin{gathered} r_0/R = 0.4, \qquad r_1/R = 0.6, \qquad \alpha\eta_1/\mu_1 = 0.01638, \\ \alpha\eta_2/\mu_2 = 0.2195, \qquad k_1/\mu_1 = 0.00165, \qquad k_2/\mu_2 = 0.0007. \end{gathered} \tag{2.7}$$

If we increase the rotation velocity even further then, at some time $t = t_1'$, when the plasticity condition $\sigma_{r\varphi}|_{r=r_1} = -k_1$ is satisfied, the viscoplastic flow will begin in the basic material as well.

In order for the flow to be possible only in the layer with the lubricant and the basic material to deform only elastically, it is necessary to rotate the cylinder with the velocity not exceeding the value

$$\omega(r_0,t_1) < \widetilde{\omega} = F\big(-k_1r_1^2, r_0, r_1\big). \tag{2.8}$$

If (2.8) is not satisfied then, starting from the time $t = t_1'$, in the basic material the region of viscoplastic flow $r_1 \le r \le x_2(t)$ will develop. In this case, integrating the equilibrium equations and using the conditions of continuity of the displacements and velocities on the surfaces $r = x_2(t)$ and $r = r_1$, we infer that

$$\theta = A(c_2), \qquad \omega = 0$$

in the domain $x_2(t) \le r \le R$;

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_1}(k_1 + c_2/r^2), \qquad p_{r\varphi} = \frac{t-t_1'}{\eta_1}\,(k_1 + c_2/r^2), \qquad c_2 = -k_1x_2^2,$$

$$\theta = \big(t - t_1'\big)F_1(c_2, r, x_2) + A(c_2), \qquad \omega = F(c_2, r, x_2),$$

$$F_1(c_1, r, x_2) = \frac{2}{\eta_1}\left(k_1 \ln\frac{r}{x_2} + \frac{c_1}{2}\big(1/x_2^2 - 1/r^2\big)\right)$$

in the domain $r_1 \le r \le x_2(t)$; and

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_2}(k_2 + c_2/r^2), \qquad p_{r\varphi} = \frac{t}{\eta_2}(k_2 + c_2/r^2),$$

$$\theta = \big(t - t_1'\big)F_1(c_2, r_1, x_2) + t_2'F(c_2, r, r_1) + A_1(c_2), \qquad \omega = F_1(c_2, r_1, x_2) + F(c_2, r, r_1)$$

in the domain $r_0 \le r \le r_1$.

The location of the elastoplastic boundary $x_2(t)$ is determined by the equation

$$\omega(r_0, t_2) = F_1(c_2, r_1, x_2) + F(c_2, r_0, r_1).$$

Let us consider how the stressed-deformed state changes if the rigid cylinder (starting from time $t = t_2'$) will rotate in the opposite direction. At first, the stresses will diminish in absolute values. At the moment of complete unloading when the stress meets $\sigma_{r\varphi} = 0$ throughout the entire deformation region $r_0 \le r \le R$, the following will hold:

$$\theta = 0, \qquad \omega = 0$$

in the domain $x_2\big(t_2'\big) \le r \le R$;

$$\varepsilon^p_{r\varphi} = 0, \qquad p_{r\varphi} = \frac{t_2' - t_1'}{\eta_1}\,(k_1 + c_3/r^2), \qquad c_3 = -k_1x_2^2\big(t_2'\big),$$

$$\theta = \big(t_2' - t_1'\big)F_1(c_3, r, x_2), \qquad \omega = 0$$

in the domain $r_1 \le r \le x_2\big(t_2'\big)$; and

$$\varepsilon^p_{r\varphi} = 0, \qquad p_{r\varphi} = \frac{t_2'}{\eta_2}\,(k_2 + c_3/r^2),$$

$$\theta = \big(t_2' - t_1'\big)F_1(c_3, r_1, x_2) + t_2'F(c_3, r, r_1), \qquad \omega = 0$$

in the domain $r_0 \le r \le r_1$.

Then the stresses increase, and, at time $t = t_3'$, the plasticity condition $\sigma_{r\varphi}|_{r=r_0} = -k_2$ holds on the surface $r = r_0$. At this moment, the value of the rotation angle of the internal rigid cylinder is

$$\theta_2 = \theta\big(r_0, t_3'\big) = \big(t_2' - t_1'\big)F_1(c_3, r_1, x_2) + t_2'F(c_3, r, r_1) + A(c_4), \qquad c_4 = k_2r_0^2.$$

From time $t = t_3'$, some region develops of the secondary viscoplastic flow, $r_0 \le r \le x_3$. Integrating the equations of equilibrium, we find for $\theta(r,t)$ and $\omega(r,t)$ that

$$\theta = A(c_5), \qquad c_5 = k_2x_3^2, \qquad \omega = 0$$

in the domain $x_2(t_2') \le r \le R$;

$$\theta = \left(t_2' - t_1'\right)F_1(c_3, r, x_2) + A(c_5), \qquad \omega = 0$$

in the domain $r_1 \le r \le x_2(t_2')$;

$$\theta = (t_2' - t_1')F_1(c_3, r_1, x_2) + t_2'F(c_3, r, r_1) + A_1(c_5), \qquad \omega = 0$$

in the domain $x_3(t) \le r \le r_1$; and

$$\theta = \left(t_2' - t_1'\right)F_1(c_3, r_1, x_2) + t_2'F(c_3, r, r_1) + tF(-c_5, x_3, r) + A_1(c_5), \qquad \omega = F(-c_5, x_3, r)$$

in the domain $r_0 \le r \le x_3(t)$.

To determine the boundary $x_3(t)$ of the viscoplastic flow region, it is necessary to specify the angular velocity on the internal rigid cylinder at each time t_3. Then we obtain $\omega(r_0, t_3) = F(-c_5, x_3, r_0)$. As it follows from the solution of this equation, the boundary $x_3(t)$ will eventually reach the external surface of the layer $r = r_1$. With the increase of the rotation velocity of the internal cylinder, the irreversible deformations will accumulate in the region $r_0 \le r \le x_3 = r_1$; however, the region itself will not become larger. When the stressed state comes out to the fluidity surface (1.3) (time t_4') then the plastic flow starts in the basic material $r_1 \le r \le x_4$ as well. In the regions $x_2(t_2') \le r \le R$ and $x_4 \le r \le x_2(t_2')$, there hold the same dependencies as before, where instead of c_5 we must insert $c_6 = k_1 x_4^2$. In the viscoplastic flow regions we find that

$$p_{r\varphi} = \frac{t - t_4'}{\eta_2}\,(c_6/r^2 - k_2) + \frac{t_2' - t_1'}{\eta_1}\,(k_1 + c_3/r^2),$$
$$\theta = \left(t_2' - t_1'\right)F_1(c_3, r, x_2) - \left(t - t_4'\right)F_1(-c_6, r, x_4) + A(c_6)$$

in the domain $r_1 \le r \le x_4$;

$$\theta = \left(t_2' - t_1'\right)F_1(c_3, r_1, x_2) - \left(t - t_4'\right)F_1(-c_6, r_1, x_4) - tF(-c_6, r, r_1) + t_2'F(c_3, r, r_1) + A_1(c_6)$$

in the domain $r_0 \le r \le r_1$. In this case, for the elastoplastic boundary $x_4(t)$, the following holds:

$$\omega(r_0, t_4) = F_1(-c_6, x_4, r_1) + F(-c_6, r_1, r_0).$$

If we further increase the rotation angle of the internal cylinder then, at time $t = t_5'$, the boundary $x_4(t)$ will reach the surface $r = x_2$, and the plastic domain will develop further. We will obtain in this case:

$$\theta = A(c_6)$$

in the domain $x_4(t) \le r \le R$;

$$\theta = \left(t_5' - t\right)F_1(-c_6, r, x_4) + A(c_6)$$

in the domain $x_2(t_2') \le r \le x_4(t)$;

$$\theta = \left(t_5' - t\right)F_1(-c_6, x_2, x_4) + \left(t_2' - t_1'\right)F_1(c_3, r_1, x_2) - \left(t - t_4'\right)F_1(-c_6, r, x_2) + A(c_6)$$

in the domain $r_1 \le r \le x_2(t_2')$; and

$$\theta = \left(t_5' - t\right)F_1(-c_6, x_2, x_4) + \left(t_2' - t_1'\right)F_1(c_3, r_1, x_2)$$
$$- \left(t - t_4'\right)F_1(-c_6, r_1, x_2) + t_2'F(c_3, r, r_1) - tF(-c_6, r, x_3) + A_1(c_6)$$

in the domain $r_0 \le r \le r_1$.

In Fig. 2, the distribution is shown of the rotation angle at time t_0 of the beginning of the plastic flow near the surface $r = r_0$, at $t_0 < t_1 < t_1^*$, at time t_1^* when the boundary $x_1(t)$ reaches the surface $r = r_1$, at t_2', at time $t_3' < t_3^* < t_4'$, and at time $t_4 > t_4'$.

In an analogous fashion, we obtain the solution of the problem in the case when the external cylinder is rotating, whereas the internal is rigidly fixed. Then, under reversible deformation we find for the function θ that

$$\theta = A_2(c), \qquad A_2(c) = \frac{c}{2\mu_1}(1/r_1^2 - 1/r^2) + \frac{c}{2\mu_2}\,(1/r_0^2 - 1/r_1^2)$$

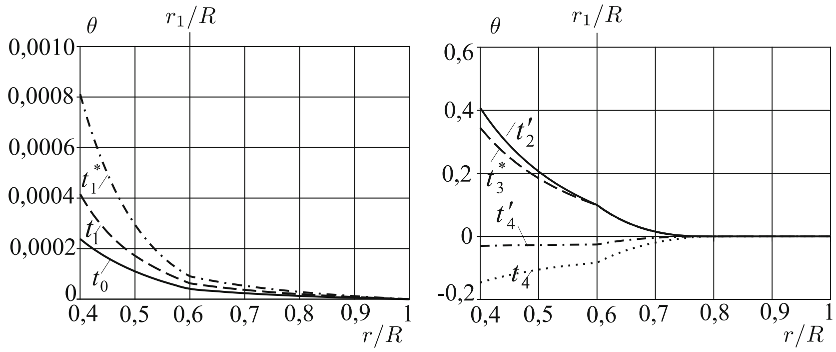

Fig. 2. Distribution of the rotation angle in dependence on the radius under rotation of the internal rigid cylinder

in the domain $r_1 \le r \le R$ and

$$\theta = A_3(c), \qquad A_3(c) = \frac{c}{2\mu_2}\,(1/r_0^2 - 1/r^2)$$

in the layer $r_0 \le r \le r_1$.

The initial parameter $\theta_0 = \theta(R, t_0)$ for the plastic flow is determined by (2.3). Under conditions of the developing viscoplastic flow, we have

$$\theta = tF(-c_1, r_0, x_1) + A_2(c_1), \qquad \omega = F(-c_1, r_0, x_1)$$

in the elastic region $r_1 \le r \le R$;

$$\theta = tF(-c_1, r_0, x_1) + A_3(c_1), \qquad \omega = F(-c_1, r_0, x_1)$$

in the elastic region $x_1 \le r \le r_1$; and

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_2}(c_1/r^2 - k_2), \qquad p_{r\varphi} = \frac{t}{\eta_2}(c_1/r^2 - k_2),$$
$$c_1 = k_2 x_1^2, \qquad \theta = tF(-c_1, r_0, r) + A_3(c_1), \qquad \omega = F(-c_1, r_0, r)$$

in the region of the viscoplastic flow $r_0 \le r \le x_1$. The moving boundary $x_1(t)$ can be found by solving the equation

$$\omega(R, t_1) = F(-c_1, r_0, x_1). \tag{2.9}$$

It follows from the comparison of (2.6) and (2.9) that, despite the difference in displacements and velocities, the viscoplastic flow domain is developing under the rotation of the external cylinder in the same way as in the case of rotation of the internal one.

At time t_1^* the boundary $x_1(t)$ reaches the surface $r = r_1$. Under further increase of the rotation velocity of the internal cylinder, the irreversible deformations are accumulated in the domain $r_0 \le r \le r_1$. The following will hold for $\theta(r,t)$ and $\omega(r,t)$:

$$\theta = tF(-c_1, r_0, r_1) + A_2(c_1), \qquad \omega = F(-c_1, r_0, r_1)$$

in the elastic region $r_1 \le r \le R$ and

$$\theta = tF(-c_1, r_0, r) + A_3(c_1), \qquad \omega = F(-c_1, r_0, r)$$

in the domain of viscoplastic flow $r_0 \le r \le r_1$.

When the angular velocity of the external cylinder becomes equal to $\widetilde{\omega}$, the plastic flow will begin in the basic material as well. We have in this case:

$$\theta = tF(-c_2, r_0, r_1) + \big(t - t_1'\big)F_1(-c_2, r_1, x_2) + A_2(c_2), \qquad \omega = F(-c_2, r_0, r_1) + F_1(-c_2, r_1, x_2)$$

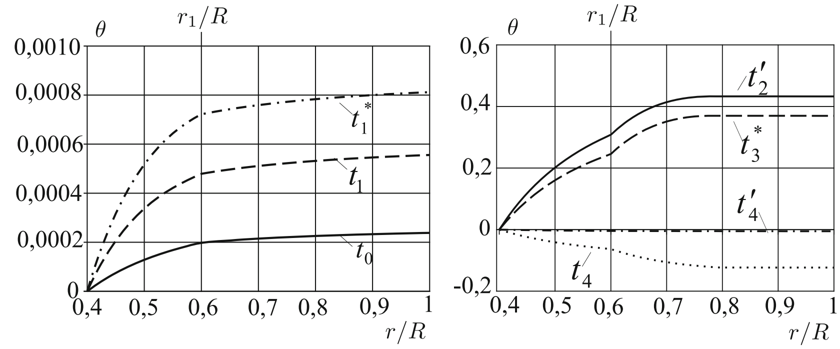

Fig. 3. Distribution of the rotation angle in dependence on the radius under the rotation of the external rigid cylinder

in the elastic region $x_2 \le r \le R$;

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_1}(c_2/r^2 - k_1), \qquad p_{r\varphi} = \frac{t - t_1'}{\eta_1}\,(c_2/r^2 - k_1),$$

$$\theta = tF(-c_2, r_0, r_1) + (t - t_1')F_1(-c_2, r_1, r) + A_2(c_2), \qquad \omega = F(-c_2, r_0, r_1) + F_1(-c_2, r_1, r)$$

in the domain of the viscoplastic flow $r_1 \le r \le x_2$; and

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_2}(c_2/r^2 - k_2), \qquad p_{r\varphi} = \frac{t}{\eta_2}(c_2/r^2 - k_2),$$

$$c_2 = k_1 x_2^2, \qquad \theta = tF(-c_2, r_0, r) + A_3(c_2), \qquad \omega = F(-c_2, r_0, r)$$

in the domain of the viscoplastic flow $r_0 \le r \le r_1$.

When the external cylinder rotates in the opposite direction, there are observed the same effects as when the internal cylinder does. In Fig. 3, there is given the distribution of the rotation angle during the entire deformation process.

3. THE LUBRICANT LAYER IS LOCATED NEXT TO THE EXTERNAL RIGID CYLINDER

Let us consider the case when the layer of a non-Newtonian lubricant is located next to the external rigid surface; i.e., it occupies the domain $r_1 \le r \le R$, whereas the deformation takes place due to rotation of the internal rigid surface. In order for the plastic flow to begin in the lubricant layer, it is necessary that $\sqrt{k_1/k_2} > r_1/r_0$; otherwise, the viscoplastic flow would develop in the basic material, whereas the layer would deform elastically. Let the geometric dimensions ensure the fulfilment of the above inequality. Then, in the conditions of elastic equilibrium, the displacement field is determined by the formulas $u_r = r(1 - \cos\theta)$, $u_\varphi = r\sin\theta$ and

$$\theta = A_4(c), \qquad A_4(c) = \frac{c}{2\mu_2}\,(1/R^2 - 1/r^2)$$

in the layer $r_1 \le r \le R$ and

$$\theta = A_5(c), \qquad A_5(c) = \frac{c}{2\mu_2}\,\big(1/R^2 - 1/r_1^2\big) + \frac{c}{2\mu_1}\,\big(1/r_1^2 - 1/r^2\big)$$

in the domain $r_0 \le r \le r_1$.

Using the plasticity condition written as $\sigma_{r\varphi}|_{r=r_1} = -k_2$, we find the rotation angle of the internal rigid cylinder under which the plastic flow starts in the layer

$$\theta_0 = \frac{k_2}{2\mu_2}\big(1 - r_1^2/R^2\big) + \frac{k_2}{2\mu_1}\big(r_1^2/r_0^2 - 1\big). \tag{3.1}$$

Under further rotation of the internal cylinder, there develops a domain of viscoplastic flow $r_1 \le r \le x_1(t)$. In this case, integrating the equilibrium equations and using the continuity condition for displacements, velocities, and stresses, we have the following:

$$\sigma_{r\varphi} = c_1/r^2, \qquad \omega = 0, \qquad c_1 = c(t_1), \qquad \theta = A_4(c_1)$$

in the elastic domain $x_1 \le r \le R$;

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_2}(k_2 + c_1/r^2), \qquad p_{r\varphi} = \frac{t}{\eta_2}(k_2 + c_1/r^2),$$
$$c_1 = -k_2 x_1^2, \qquad \theta = tF(c_1, r, x_1) + A_4(c_1), \qquad \omega = F(c_1, r, x_1)$$

in the domain of the viscoplastic flow $r_1 \le r \le x_1$; and

$$\theta = tF(c_1, r_1, x_1) + A_5(c_1), \qquad \omega = F(c_1, r_1, x_1)$$

in the elastic domain $r_0 \le r \le r_1$.

The boundary of the viscoplastic flow domain is found by solving the equation

$$\omega(r_0, t_1) = F(c_1, r_1, x_1).$$

We assume that the location and the size of the lubricant layer are chosen so that the boundary $x_1(t)$ reaches the surface of the external rigid cylinder prior to the viscoplastic flow starting in the basic material; i.e., there holds the condition

$$F(-k_2R^2, r_1, R) < F\big(-k_1r_0^2, r_1, r_0\sqrt{k_1/k_2}\,\big).$$

Then the two regions remain in the material: the region of elastic deformation $r_0 \le r \le r_1$ and the region $r_1 \le r \le R$ in which the irreversible deformations continue to accumulate. In this case, the functions $\theta(r,t)$ and $\omega(r,t)$ have the form

$$\theta = tF(c_1, r, R) + A_4(c_1), \qquad \omega = F(c_1, r, R)$$

in the region of viscoplastic flow $r_1 \le r \le R$;

$$\theta = tF(c_1, r_1, R) + A_5(c_1), \qquad \omega = F(c_1, r_1, R)$$

in the elastic region $r_0 \le r \le r_1$.

The limit value of the rotation velocity of the internal cylinder under which the basic material deforms elastically can be obtained by $\widetilde{\omega} = F(-k_1r_0^2, r_1, R)$. If the rotation velocity of the cylinder exceeds this value then, in the neighborhood of the internal rigid wall, beginning from time t_1', there will develop a domain of viscoplastic flow $r_0 \le r \le x_2$. In the domains $r_1 \le r \le R$ and $x_2 \le r \le r_1$, there will hold the same relations as before; in the domain $r_0 \le r \le x_2$ we find

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_1}(k_1 + c_2/r^2), \qquad p_{r\varphi} = \frac{t}{\eta_1}(k_1 + c_2/r^2), \qquad c_2 = -k_1x_2^2,$$
$$\theta = \big(t - t_1'\big)F_1(c_2, r, x_2) + tF(c_2, r_1, R) + A_5(c_2), \qquad \omega = F_1(c_2, r, x_2) + F(c_2, r_1, R).$$

For the elastoplastic boundary $x_2(t)$ we will obtain the equation

$$\omega(r_0, t_2) = F_1(c_2, r_0, x_2) + F(c_2, r_1, R).$$

Let the internal rigid cylinder rotate in the opposite direction starting from time t_2'. Let us assume that, at this time, the entire lubricant layer is plastically flowing; whereas, the basic material is deforming only elastically; i.e., the rotation velocity of the rigid cylinder does not exceed $\widetilde{\omega}$. At time of complete unloading, we have:

$$\theta = t_2'F(c_3, r, R), \qquad \omega = 0, \qquad c_3 = c\big(t_2'\big)$$

in the region with unchanging plastic deformations $r_1 \le r \le R$;

$$\theta = t_2'F(c_3, r_1, R), \qquad \omega = 0$$

in the elastic region $r_0 \le r \le r_1$.

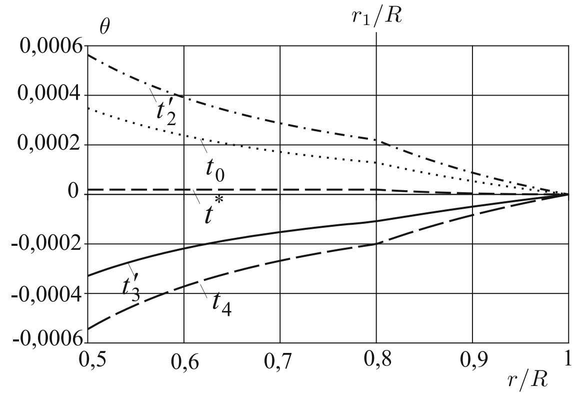

Fig. 4. Distribution of the rotation angle in dependence on the radius under the rotation of the internal rigid cylinder

The rotation angle on which the internal cylinder is to be rotated in order for the plastic flow to start again on the surface $r = r_1$ is given by the value

$$\theta_2 = \theta\bigl(r_0, t_3'\bigr) = t_2' F(c_4, r_1, R), \qquad c_4 = k_2 r_1^2.$$

Integrating the equilibrium equations in the three domains, we infer

$$\varepsilon^p_{r\varphi} = 0, \qquad p_{r\varphi} = \frac{t_2'}{\eta_2}\,(k_2 + c_3/r^2), \qquad \theta = t_2' F(c_3, r, R) + A_4(c_5), \qquad \omega = 0, \qquad c_5 = k_2 x_3^2$$

in the domain with unchanging plastic deformations $x_3 \le r \le R$;

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_2}(c_5/r^2 - k_2), \qquad p_{r\varphi} = \frac{t}{\eta_2}(c_5/r^2 - k_2) + \frac{t_2'}{\eta_2}\,(k_2 + c_3/r^2),$$

$$\theta = t_2' F(c_3, r, R) - tF(-c_5, r, x_3) + A_4(c_5), \qquad \omega = F(-c_5, x_3, r)$$

in the domain of the viscoplastic flow $r_1 \le r \le x_3$; and

$$\theta = t_2' F(c_3, r_1, R) - tF(-c_5, r_1, x_3) + A_5(c_5), \qquad \omega = F(-c_5, x_3, r_1)$$

in the elastic domain $r_0 \le r \le r_1$.

In time, the boundary $x_3(t)$ reaches the external rigid surface $r = R$. Then irreversible deformations accumulate in the layer $r_1 \le r \le R$. When the rotation velocity of the internal rigid cylinder becomes equal to $\widehat{\omega} = F(-k_1 r_0^2, R, r_1)$, a plastic flow region will develop near the internal surface $r = r_0$. The absolute value $|\widehat{\omega}|$ of the angular velocity of the internal cylinder necessary for the beginning of plastic flow coincides with $\widetilde{\omega}$ for the first turn. The graphs of the rotation angle at various times are given in Fig. 4 ($r_0/R = 0.5$ and $r_1/R = 0.8$).

Now, let the external cylinder rotate, whereas let the internal one be rigidly fixed. For reversible deformation, using the adhesion condition on the motionless cylinder, we obtain

$$\theta = A_6(c), \qquad A_6(c) = \frac{c}{2\mu_2}\,\bigl(1/r_1^2 - 1/r^2\bigr) + \frac{c}{2\mu_1}\,\bigl(1/r_0^2 - 1/r_1^2\bigr)$$

in the layer $r_1 \le r \le R$;

$$\theta = A_7(c), \qquad A_7(c) = \frac{c}{2\mu_1}\,(1/r_0^2 - 1/r^2)$$

in the domain $r_0 \le r \le r_1$.

The value of the rotation angle of the external rigid cylinder, under which the plasticity condition holds on the internal surface of the layer $r = r_1$ and the plastic flow starts, is determined by (3.1). The developing region of the viscoplastic flow will be bounded by the surfaces $r = r_1$ and $r = x_1(t)$. In the conditions of quasistatic approximation, we find:

$$\theta = tF(-c_1, r_1, x_1) + A_6(c_1), \qquad \omega = F(-c_1, r_1, x_1)$$

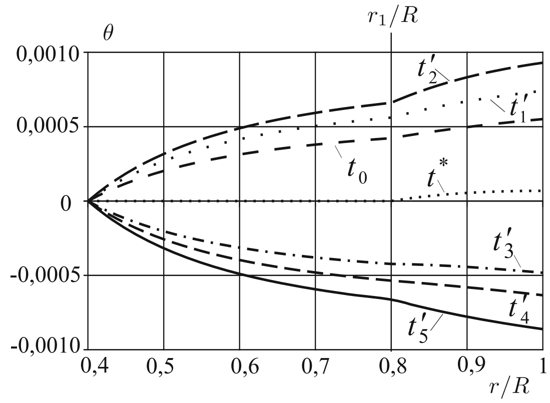

Fig. 5. Distribution of the rotation angle in dependence on the radius under the rotation of the external rigid cylinder

in the elastic domain $x_1 \le r \le R$;

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_2}\,(c_1/r^2 - k_2), \qquad p_{r\varphi} = \frac{t}{\eta_2}\,(c_1/r^2 - k_2), \qquad c_1 = k_2 x_1^2,$$

$$\theta = tF(-c_1, r_1, r) + A_6(c_1), \qquad \omega = F(-c_1, r_1, r)$$

in the region of the viscoplastic flow $r_1 \le r \le x_1$; and

$$\theta = A_7(c_1), \qquad \omega = 0$$

in the elastic domain $r_0 \le r \le r_1$.

We obtain the following equation to determine the elastoplastic boundary:

$$\omega(R, t_1) = F(-c_1, r_1, x_1).$$

Let the geometric dimensions be selected so that the plastic flow in the basic material begins when the boundary $x_1(t)$ has not yet reached the external surface $r = R$. In this case, beginning from time $t = t_1'$, the equilibrium equations should be integrated in the four domains. Using the continuity of velocities and displacements, we infer

$$\theta = tF(-c_1, r_1, x_1) + (t - t_1)F_1(-c_1, r_0, x_2) + A_6(c_1), \qquad \omega = F(-c_1, r_1, x_1) + F_1(-c_1, r_0, x_2)$$

in the elastic domain $x_1 \le r \le R$;

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_2}(c_1/r^2 - k_2), \qquad p_{r\varphi} = \frac{t}{\eta_2}(c_1/r^2 - k_2), \qquad c_1 = k_2 x_1^2,$$

$$\theta = tF(-c_1, r_1, r) + (t - t_1)F_1(-c_1, r_0, x_2) + A_6(c_1), \qquad \omega = F(-c_1, r_1, r) + F_1(-c_1, r_0, x_2)$$

in the domain of the viscoplastic flow $r_1 \le r \le x_1$;

$$\theta = (t - t_1)F_1(-c_1, r_0, x_2) + A_7(c_1), \qquad \omega = F_1(-c_1, r_0, x_2)$$

in the elastic domain $x_2 \le r \le r_1$; and

$$\varepsilon^p_{r\varphi} = \frac{1}{\eta_1}(c_1/r^2 - k_1), \qquad p_{r\varphi} = \frac{t - t_1'}{\eta_1}\,(c_1/r^2 - k_1), \qquad c_1 = k_2 x_1^2 = k_1 x_2^2,$$

$$\theta = (t - t_1)F_1(-c_1, r_0, r) + A_7(c_1), \qquad \omega = F_1(-c_1, r_0, r)$$

in the domain of the viscoplastic flow $r_0 \le r \le x_2$.

At time $t = t_2'$ the boundary $x_1(t)$ will reach the external surface $r = R$. In addition, in the domains $r_1 \le r \le R$, $x_2 \le r \le r_1$, and $r_0 \le r \le x_2$, the previous dependencies hold. As in the above problems, we consider the deformation of the material under the rigid cylinder rotation in the opposite direction. The distribution of the rotation angle is shown in Fig. 5 ($r_0/R = 0.4$ and $r_1/R = 0.8$) at the times of the beginning of flow in the layer, $t = t_0$, and of the flow beginning in the basic material, $t = t_1'$; at time t_2'

when the boundary x_1 reaches the surface $r = R$ (from this time on, the external cylinder rotates in the opposite direction); at time t_* of complete unloading; at time t'_3 of the beginning of the secondary flow for $r = r_1$; at time t'_4 of the beginning of the secondary flow for $r = r_0$; and at time t'_5 when the boundaries of the new regions of the viscoplastic flow reach the surfaces $r = R$ and $r = x_2(t'_2)$, correspondingly.

REFERENCES

1. E. N. Lee, "Elastic-Plastic Deformation at Finite Strains," Trans. ASME J. Appl. Mech. **36** (1), 1–6 (1969).
2. V. I. Kondaurov, "On Equations of a Viscoelastoplastic Medium with Finite Deformations," Prikl. Mekh. i Tekhn. Fiz. No. 4, 133–139 (1982).
3. S. Nemat-Nasser, "On Finite Deformation Elasto-Plasticity," Internat. J. Solids Structures **18** (10), 857–872 (1982).
4. V. I. Levitas, *Large Elastoplastic Deformations of Solids under High Pressure* (Naukova Dumka, Kiev, 1987) [in Russian].
5. G. I. Bykovtsev and A. V. Shitikov, "Finite Deformations of Elastoplastic Media," Dokl. Akad. Nauk **311** (1), 59–62 (1990).
6. V. P. Myasnikov, "Equations of Motion of Elastoplastic Solids under Heavy Strains," Vestnik Dal'nevost. Otdel. Ross. Akad. Nauk No. 4, 8–13 (1996).
7. A. A. Rogovoi, "Constitutive Relations for Finite Elastic-Inelastic Strains," Prikl. Mekh. i Tekhn. Fiz. **46** (5), 138–149 (2005) [J. Appl. Mech. Tech. Phys. **46** (5), 730–739 (2005)].
8. A. D. Chernyshev, "Defining Equations of an Elastoplastic Solid under Finite Strains," Izv. Ross. Akad. Nauk Mekh. Tverd. Tela No. 1, 120–128 (2000).
9. A. A. Pozdeev, P. V. Trusov, and Yu. I. Nyashin, *Large Elastoplastic Deformations: Theory, Algorithms, and Applications* (Nauka, Moscow, 1986) [in Russian].
10. A. A. Burenin, G. V. Bykovtsev, and L. V. Kovtanyuk, "A Simple Model of Finite Strains in an Elastoplastic Medium," Dokl. Akad. Nauk **347** (2), 199–201 (1996) [Phys. Dokl. **41** (3), 127–129 (1996)].
11. A. A. Burenin and L. V. Kovtanyuk, *Elastic Effects under Intensive Irreversible Deformation* (Dal'nevost. Gos. Tekhn. Univ., Vladivostok, 2011) [in Russian].
12. B. D. Annin and G. P. Cherepanov, *An Elastoplastic Problem* (Nauka, Novosibirsk, 1983) [in Russian].
13. A. A. Burenin, L. V. Kovtanyuk, and M. V. Polonik, "Possibility of a Secondary Plastic Flow under General Unloading of an Elastoplastic Medium," Dokl. Akad. Nauk **375** (6), 767–769 (2000).
14. A. A. Burenin, L. V. Kovtanyuk, and M. V. Polonik, "The Formation of a One-Dimensional Residual Stress Field in the Neighborhood of a Cylindrical Defect in the Continuity of an Elastoplastic Medium," Prikl. Mat. i Mekh. **67** (2), 316–325 (2003) [J. Appl. Math. Mech. **67** (2), 283–292 (2003)].
15. L. V. Kovtanyuk, "About Punching Shear of a Viscoelastoplastic Material Through a Solid Round Cylindrical Matrix," Dokl. Akad. Nauk **400** (6), 764–767 (2005).
16. A. A. Burenin, L. V. Kovtanyuk, and A. S. Ustinova, "On Accounting for the Elastic Properties of a Non-Newtonian Viscosimetric Flow," Prikl. Mekh.i Tekhn. Fiz. **49** (2), 143–151 (2008).
17. V. A. Znamenskii and D. D. Ivlev, "On Equations of an Elastoplastic Solid under Piecewise Linear Potentials," Izv. Akad. Nauk SSSR, Mekh. i Mashinostroenie No. 6, 114–118 (1963).