2017 Fourth International Conference on Mathematics and Computers in Sciences and in Industry

Applications of Multi-Physics Modelling for Simulations of Thermo-Elastic-Plastic Materials

Evgenii Murashkin
Ishlinsky Institute for Problems in Mechanics
Russian Academy of Science
Moscow, Russia, 119526
101 Bldg 1 Vernadsky Ave
Email: evmurashkin@gmail.com

Evgeniy Dats
Vladivostok State University of
Economics and Service
Vladivostok, Russia, 690014
41 Gogolya str.
Email: dats@dvo.ru

***Abstract*—The intensive deformation processes of elastic-plastic material are accompanied by heating in the plastic flow domain. The aim of this study is to investigate the coupling of the plastic deformations and temperature during rapid heating. Numerical-analytical algorithm for the coupled thermoelastoplasticity problem is proposed. The boundary value problem of a continuous elastic-plastic ball heating is considered. The analytical solutions for the stress-strain state parameters were earlier obtained under the temperature stresses theory frameworks and piecewise yield criteria. The simultaneous existence of the unloading and plastic flow domains under heating process was shown and analyzed. The nonstationary heat conduction equation has the simplest form in a spherical coordinate system inside unloading domains. Otherwise, the source terms are aroused in the heat conduction equation inside the plastic flow domain.**

I. Introduction

The problem of the stress-strain state computation under plastic flow conditions is more sophisticated under intensive thermal action. The temperature gradient is similar to the mass forces and the change of the temperature field can be described as a certain mass force action distributed by material volume. In depth studies of plastic flow under the external action given on the boundary surfaces are discussed in [1], [2], [3], [4]. The residual stresses are the essential factor in most technological processes including natural phenomena and additive manufacturing. The problems concerning residual stresses computations in the frameworks of the large elastic-plastic deformations are discussed in [1], [3], [4]. Some results were presented in studies concerning residual stress calculations within the frameworks of the surface growth theory to problems in geomechanics [5] and additive manufacturing technologies [6], [7]. There are analytical solutions obtained under the Tresca yield criterion [8], [9], [10], [11], [12], [13], [2], [14], [15], [16]. In some cases, a model with hardening was used under high values of plastic deformations provoking the increasing yield strength. At contrary, it is necessary to take into account the yield strength decreasing during the heating process according to the experimental data.

Plastic flow is begun and irreversible deformations are accumulated under the stress state reaching the yield surface. A general algorithm in the frameworks of the finite element method is based on the elastic-plastic matrix constructing at each time for the increments of stresses and total deformations (see in details [17], [18], [19]). However, the this procedure using can lead to the non-uniqueness of the numerical solution or misconvergence in the iterative processes accompanying the numerical solutions of nonlinear problems. An alternative way to solve elastic-plastic boundary value problems is the numerical solution of the resulting system consisting of the equations of equilibrium and the yield criterion. The resulting differential equations can be reduced to non-linear algebraic equations by the finite-difference approximations. Such problem is solved by the conventional method of initial iterations. A particular feature of this method is the simultaneous calculation of the displacement vector (stresses) and the Lagrange (undetermined) multiplier.

The solutions of the simplest one-dimensional boundary value problems under piecewise linear yield criteria should be fulfill the following requirements:

1) Stresses, deformations and displacements are continuous at the elastic-plastic boundaries;
2) Plastic strain increments and deviatoric stress tensor have the same principal directions (Associated flow rule, von Mises principle).

The first requirement is satisfied automatically by the equilibrium equations integrating. The fulfillment of the second requirement depends on the convergence of the numerical solution and is the main criterion for the reliability. Thus in the present study we consider the numerical solution algorithm for the thermal stress problem taking into account the plastic properties.

The equilibrium equations integrating in terms of the displacement vector give us the analytical solution in the framework of the perfect plasticity theory. The Tresca yield criterion accurately describes the elastic-plastic behaviour of a material under shear deformations. Another piecewise linear yield criterion is the Ishlinsky-Ivlev (maximum reduced stress) yield criterion (see in details [12], [13], [20], [21]). The von Mises yield criterion is more preferred in the case of thermal expansion. The nonlinearity of this condition leads to the necessity of numerical integration of the equilibrium equations even in the one-dimensional case. Numerical algorithms have

978-1-5386-2820-1/17 $31.00 © 2017 IEEE
DOI 10.1109/MCSI.2017.64

a high error value for non-stationary nonuniform temperature effects calculation in consequence of the simultaneous existence of plastic flow and unloading domains and elastic-plastic boundaries motion. At present study we proposed a technique of the residual stress computing in the frameworks of the piecewise linear Tresca and Ishlinsky-Ivlev yield criteria. In the stationary thermal action case similar solutions correspond to the von Mises yield criterion ones. As shown below, the calculation process within frameworks of the piecewise linear yield criterion is simpler and faster in the non-stationary case in contrary to the numerical integration by virtue of the von Mises yield criterion.

mds
August 26, 2015

II. Governing Equations

Throughout the study the conventional Prandtle–Reuss elastic-plastic model generalised on thermal effects [22], [14] is used. The infinitesimal strain tensor d_{ij} additively splitted into two parts as follows

$$d_{ij} = e_{ij} + p_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1}$$

wherein e_{ij} and p_{ij} denote the elastic (reversible) and plastic (irreversible) deformations respectively (1); u_i is the displacements vector components; index after coma denotes the partial spatial derivative.

The Duhamel–Neumann constitutive strain-stress equations are given by

$$\sigma_{ij} = \left(\lambda e_{kk} - \alpha\left(3\lambda + 2\mu\right)T\right)\delta_{ij} + 2\mu e_{ij}. \tag{2}$$

Here σ_{ij} is the Cauchy stress tensor; T is the difference between the actual and referential temperatures; λ, μ are the Lame mudulus; α denotes the thermal expansion coefficient. It is assumed that the thermal field is known at each point of the material. In this case we can describe the stress-strain state by the thermal stresses theory [22]. The equilibrium equations can be formulated as

$$\sigma_{ij,j} = 0. \tag{3}$$

The significant stress level can cause plastic flow process. In such domains the deformation processes occur in the frameworks of perfect plasticity theory. The main thermodynamical principle of this theory is the the von Mises maximum principle (R. von Mises) [23], [24]. The main consequence of this one is geometric convexity of yield surface $f(\sigma_{ij}) = 0$ in Haigh-Westergaard stress space) and orthogonality of plastic strain increment dp_{ij} to yield surface at its smoothness points for an actual plastic flow process. The principle of geometrical orthogonality in stress space is simultaneously the main constitutive law of the mathematical theory of perfect plasticity known also as associated flow rule

$$dp_{ij} = d\xi\frac{\partial f}{\partial\sigma_{ij}}, \tag{4}$$

where $d\xi \geq 0$ is the undetermined multiplier treated as a Lagrange multiplier appearing while solving extreme problem corresponding to the maximum principle. The indeterminateness of multiplier $d\xi$ in theory of perfect plasticity is elucidated by the fact that it is not considered as a given function of the thermodynamic state variables and therefore a special constitutive equation need not be formulated.

The yield criterion can be formulated as follows: piecewise linear Tresca yield criterion [8] (maximum tangential stress one)

$$f = \max\left\{\left|\sigma_1 - \sigma_2\right|, \left|\sigma_2 - \sigma_3\right|, \left|\sigma_3 - \sigma_1\right|\right\} - 2k(T) = 0; \tag{5}$$

piecewise linear Ishlinsky–Ivlev yield criterion [12], [13] (maximum reduced stress one)

$$f = \max\left\{\left|\sigma_1 - \sigma\right|, \left|\sigma_2 - \sigma\right|, \left|\sigma_3 - \sigma\right|\right\} - \frac{4k}{3}(T) = 0; \tag{6}$$

von Mises yield criterion [25], [23] (maximum equivalent tensile stress one)

$$f = \left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(\sigma_{ji} - \frac{1}{3}\sigma_{kk}\delta_{ji}\right) - \frac{8}{3}k^2(T) = 0. \tag{7}$$

Herein, $k(T)$ is the yield stress decreasing with increasing temperature; $\sigma = (\sigma_1+\sigma_2+\sigma_3)/3$, σ_1, σ_2, σ_3 are the principal stresses. We assume that the yield stress is the linear function of the actual temperature

$$k(T) = k_0(1 - \beta(T - T_0)). \tag{8}$$

Here k_0 is the referential yield stress, β is the constitutive constant, which can be experimentally obtained.

The yield criteria (5)–(7) can be presented as surface in the stress space. In particular, the Tresca and Ishlinsky-Ivlev yield criteria are the hexagonal prisms inclined to the coordinate axes, and the von Mises one is the cylinder. The projections of the yield criteria on deviatoric plane shown on Fig. 2.

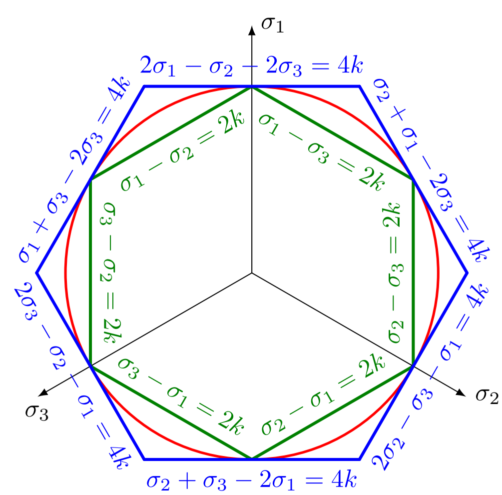

Fig. 1. Yield criteria in deviatory plane. The red circle is the von Mises yield criterion. The inscribed green hexagon is the Tresca yield criterion. The escribed blue hexagon is the Ishlinsky-Ivlev yield criterion.

Hereafter we use the von Mises yield criterion. The equations for plastic deformations increments can be derived from equations (4), (7) in follow form

$$dp_{ij} = d\xi \left(3\sigma_{ij} - \sigma_{kk}\delta_{ij}\right) \tag{9}$$

The plastic incompressibility of a material is a consequence of equation (9)

$$dp_{ii} = 0. \tag{10}$$

The plastic deformations increments are represented as follows

$$dp_{ij} = p_{ij} - p'_{ij}, \tag{11}$$

where p'_{ij} denote the plastic deformations calculated at the previous time. Thus, to solve the problem, the stress-strain state calculations are to be divided into time intervals, the number of which is determine the accuracy of the numerical solution.

For elastic deformations one can obtained according the equation (1)

$$e_{ij} = d_{ij} - p_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) - p_{ij}, \tag{12}$$

Assuming that there is a state at some referential time when $p'_{ij} = 0$ we can replace $dp_{ij} = p_{ij}$ according to equations (10), (11). Then plastic strain trace is vanished

$$p_{ii} = 0, \tag{13}$$

and total deformations are read

$$d_{ii} = e_{ii} = u_{i,i}. \tag{14}$$

The elastic deformations are derived by equations (9), (11), (12) as

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) - d\xi \left(3\sigma_{ij} - \sigma_{kk}\delta_{ij}\right) - p'_{ij} \tag{15}$$

Then one can obtained by substituting the equation (15) in (2) taking into account the equation (14)

$$\begin{gathered} \sigma_{ij} = \left(Lu_{k,k} - m\left(T - T_0\right)\right)\delta_{ij} + M(u_{i,j} + u_{j,i} - 2p'_{ij}), \\ L(d\xi) = \lambda + \frac{4\mu^2 d\xi}{(1 + 6\mu d\xi)}, \quad M(d\xi) = \frac{\mu}{(1 + 6\mu d\xi)}. \end{gathered} \tag{16}$$

Finally, the resulting differential system can be rewritten after substituting (16) into equilibrium equations (3)

$$\begin{gathered} (L + M)u_{k,ki} + u_{k,k}L_{,i} - mT_{,i} + M(u_{i,kk} - 2p'_{ik,k}) \\ +M_{,k}(u_{i,k} + u_{k,i} - 2p'_{ik}) = 0. \end{gathered} \tag{17}$$

In the case of elastic deformation process we have $d\xi = 0$, $L = \lambda$, $M = \mu$ and equations (17) are transformed to Lame equations with accumulated irreversible deformations

$$(\lambda + \mu)\, u_{k,ki} - mT_{,i} + \mu\left(u_{i,kk} - 2p'_{ik,k}\right) = 0 \tag{18}$$

Inside the plastic flow domain the equations (17) and yield criterion (7) form a system of 4 equations with respect to variables u_i, $d\xi$.

III. Boundary Value Problem Statment

Consider the one-dimensional problem of an elastic-plastic material deformation under uneven heating. The choice of this statement is due to the available analitical solutions within the frameworks of piecewise linear yield criteria. These solutions make it possible to estimate the correctness of the numerical calculations. There is an infinitely long hollow cylinder $d_{zz} = 0$ with inner and outer radii R_1 and R_2 respectively. The temperature at the inner surface of the cylinder is give by $T(R_1, t) = t$. There is no temperature change on the outer surface $T(R_2, t) = 0$. The temperature field is described by the solution of the stationary heat conduction equation under polar symmetry conditions

$$T(r,t) = \frac{t\ln(r/R_2)}{\ln(R_1/R_2)}. \tag{19}$$

The free thermal expansion conditions of at the boundary surfaces of the cylinder are assumed

$$\sigma_{rr}(R_1, t) = 0, \quad \sigma_{rr}(R_2, t) = 0. \tag{20}$$

The equations for nonzero deformations are read by

$$\begin{gathered} d_{rr} = u_{r,r} = e_{rr} + p_{rr}, \\ d_{\varphi\varphi} = r^{-1}u_r = e_{\varphi\varphi} + p_{\varphi\varphi}, \\ d_{zz} = e_{zz} + p_{zz} = 0, \end{gathered} \tag{21}$$

where u_r is the radial components of the displacements vector.

For nonzero stresses we can obtain the following equations taking into account the equations (2)

$$\begin{aligned} \sigma_{rr} &= (\lambda + 2\mu)e_{rr} + \lambda(e_{\varphi\varphi} + e_{zz}) - (3\lambda + 2\mu)\Delta, \\ \sigma_{\varphi\varphi} &= (\lambda + 2\mu)e_{\varphi\varphi} + \lambda(e_{rr} + e_{zz}) - (3\lambda + 2\mu)\Delta, \\ \sigma_{zz} &= (\lambda + 2\mu)e_{zz} + \lambda(e_{\varphi\varphi} + e_{rr}) - (3\lambda + 2\mu)\Delta. \end{aligned} \tag{22}$$

Here $\Delta = \alpha T$ denotes thermal expansion defromation. The equilibrium equation (3) and compatibility conditions in the case of polar symmetry take the form

$$\sigma_{\varphi\varphi} = (r\sigma_{rr})_{,r}, \quad d_{rr} = (rd_{\varphi\varphi})_{,r}. \tag{23}$$

Then the reversible deformations e_{rr}, $e_{\varphi\varphi}$ and stress σ_{zz} from equations (9) and (22) can be furnished by

$$\begin{gathered} e_{rr} = \frac{\left(2\lambda\mu p'_{zz} + 2\Delta M + (\Lambda + 2M)\sigma_{rr} - \Lambda\sigma_{\varphi\varphi}\right)}{4\mu(\Lambda + M)}, \\ e_{\varphi\varphi} = \frac{\left(2\lambda\mu p'_{zz} + 2\Delta M + (\Lambda + 2M)\sigma_{\varphi\varphi} - \Lambda\sigma_{rr}\right)}{4\mu(\Lambda + M)}, \\ \sigma_{zz} = \frac{((\Lambda + M - \mu)(\sigma_{rr} + \sigma_{\varphi\varphi}) - 2\mu(3\lambda + 2\mu)(p'_{zz} + \Delta))}{2(\Lambda + M)}, \end{gathered} \tag{24}$$

wherein $\Lambda(r,t) = (\lambda + 12\lambda\mu d\xi(r,t))$, $M(r,t) = (\mu + 8\mu^2 d\xi(r,t))$. Note that under condition $d\xi = 0$ the equations (24) correspond to thermoplastic deformation ($\Lambda = \lambda$, $M = \mu$).

Substitute the equations (9), (24) in (23) and derived

$$\begin{aligned}&\frac{r^2\Lambda(1+\mu\Gamma)}{4\lambda\mu}\sigma_{rr,rr}+\left(\frac{3r\Lambda(1+\mu\Gamma)}{4\lambda\mu}+\right.\\&\left.+\frac{r^2M_{,r}}{8}\left(\Gamma^2+\tfrac{3}{\mu^2}\right)\right)\sigma_{rr,r}+\\&+\frac{r}{4}(3\lambda+2\mu)\left(p'_{zz}+\Delta+\frac{\sigma_{rr}}{(3\lambda+2\mu)}\right)M_{,r}+\\&+\frac{r}{2\lambda}(3\lambda+2\mu)\Delta_{,r}+p'_{\varphi\varphi}-p'_{rr}+\\&+rp'_{\varphi\varphi,r}+\frac{r}{2}(1-\mu\Gamma)p'_{zz,r}=0,\end{aligned}\tag{25}$$

where $\Gamma(r,t)=\dfrac{1}{(\Lambda(r,t)+M(r,t))}$.

The von Mises yield criterion in the frameworks of the considering problem is formulated by

$$\begin{aligned}f=\sigma_{rr}^2+\sigma_{\varphi\varphi}^2+\sigma_{zz}^2-\sigma_{rr}\sigma_{zz}-\\-\sigma_{rr}\sigma_{\varphi\varphi}-\sigma_{zz}\sigma_{\varphi\varphi}-4k^2=0.\end{aligned}\tag{26}$$

The system of equations (25) and (26) relatively the variables σ_{rr}, $d\xi$ simulates the stress-strain state of the cylinder during the plastic flow. The solution of this system under the boundary conditions (ref free) is numerically found using the method of successive approximations. On this way the differential equations (25), (26) reduced to the algebraic equations system replacing the derivatives on the standard finite difference approximations. Thus, the material of the cylinder is divided into a finite number of nodes along the radial coordinate $r(i)=R_1+i\dfrac{(R_2-R_1)}{im}$ (i is the node number, im is the node amount). In each node a finite-difference approximation of the system of equations (25), (26) is constructed. Consider the algorithm for calculations of the stress-strain state parameters. The calculations are sequentially performed at each time step $t(j)=jdt$, using the results of the calculations in the previous step. The following conditions $p'_{ij}(r(i),0)=0$, $d\xi(r(i),0)=0$ are given at the initial time $t_0=0$. At the next time $t(j)$ the temperature field is changed and a system composed of $(im-2)$ equations (24) given in the inside nodes under boundary conditions (20) is solved. Thus, a numerical solution is constructed for σ_{rr} within the frameworks of the thermoelasticity. At each time step the condition $f>0$ (26) is tested. If it is performed at certain node we add the equation $f=0$ to equation (25) with new variable $d\xi$. If $f<0$ at a node then $d\xi=0$. The number of nodes satisfying the condition $f>0$ is increased with the plastic flow developing. The closest node to the plastic flow domain under conditions $f<0$ ($d\xi=0$) can be considered as an approximate position of the elastic-plastic boundary. The condition $d\xi>0$ at present study is used as a criterion for the numerical solution correctness inside the plastic flow domain. Therefore, the number of spatial nodes and time intervals is chosen in such a way that the parameter $d\xi$ are always positive during the calculations. Negative values of this parameter mean a high calculation error and incorrect determination of the irreversible deformation domain boundaries.

IV. Results Discussion

In this section we use the analytical solution for the Tresca and Ishlinskii-Ivlev piecewise linear yield criteria to verify the correctness of the numerical solution for the von Mises yield criterion. On the Fig. 2 the thermal stresses are shown.

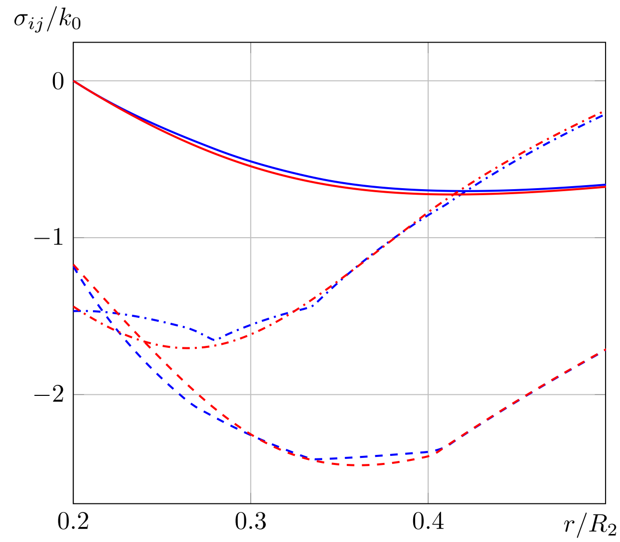

Fig. 2. Thermal stresses. Arithmetic mean (blue line) of the solutions for the Tresca and the Ishlinsky–Ivlev yield criteria. The numerical solution (red line) for the von Mises yield criterion. $R_1/R_2=0.2$

Note that the boundaries of the plastic flow domains for both solutions coincide with an accuracy of 99.97 percent. Calculations show that this accuracy has the maximum value in the case when the undetermined multiplier is positive at each point inside the irreversible deformation domain at any time. A further increasing in the number of nodes along the spatial and time coordinates does not lead to a change in the position of the elastic-plastic boundary. Thus, we can conclude that the resulting positive value of the multiplier $d\xi$ at each time step corresponds to the correctness of the assumption that at a given time step the plastic flow is present at the *elastic* points where stress exceeds the yield stress.

V. Conclusions

The conventional Prandtl–Reuss model generalized by the thermal effects under the von Mises yield criterion have been used. It is assumed that the material is deformed without hardening and the yield stress depends on the temperature. It is shown that the system of equations for the stress-strain state calculations can be reduced to algebraic equations system by means of finite-difference approximations. It is established that criterion for the numerical solution correctness is the condition of Lagrange multiplier positivity.

Acknowledgment

The authors would like to thank Professor A. Burenin for his helpful comments, suggestions and encouragement. The research was supported by RSF (project No. 17-19-01257)

REFERENCES

[1] Burenin, A. A., Kovtanyuk, L. V., and Murashkin, E. V., 2006. "On the residual stresses in the vicinity of a cylindrical discontinuity in a viscoelastoplastic material". *Journal of Applied Mechanics and Technical Physics,* ***47***(2), pp. 241–248.

[2] Kovtanyuk, L. V., and Murashkin, E. V., 2009. "Onset of residual stress fields near solitary spherical inclusions in a perfectly elastoplastic medium". *Mechanics of Solids,* ***44***(1), pp. 78–87.

[3] Bazhin, A. A., and Murashkin, E. V., 2012. "Creep and stress relaxation in the vicinity of a micropore under the conditions of hydrostatic loading and unloading". *Doklady Physics,* ***57***(8), pp. 294–296.

[4] Burenin, A. A., Kovtanyuk, L. V., and Murashkin, E. V., 2012. "Strengthening of materials by intensive hydrostatic compression pretreatment". *Mechanics of Solids,* ***47***(6), pp. 665–670.

[5] Manzhirov, A. V., and Parshin, D. A., 2016. "Accretion of spherical viscoelastic objects under self-gravity". *Lecture Notes in Engineering and Computer Science,* ***2224***(1), pp. 1131–1135.

[6] Manzhirov, A. V., 2013. "Mechanics of growing solids and phase transitions". *Key Engineering Materials,* ***535***, pp. 89–93.

[7] Manzhirov, A. V., and Parshin, D. A., 2015. "Influence of the erection regime on the stress state of a viscoelastic arched structure erected by an additive technology under the force of gravity". *Mechanics of Solids,* ***50***(6), pp. 657–675.

[8] Tresca, H., 1864. "Mémoire sur l'écoulement des corps solides soumis à de fortes pressions". *Comptes Rendus de l'Académie des Sciences,* ***59***, p. 754.

[9] Mack, W., and Gamer, U., 1985. "Die spannungsverteilung in der elastisch-plastischen kreisscheibe infolge einer kreisflachenformigen warmequelle". *Forschung im Ingenieurwesen,* ***5***, pp. 160–164.

[10] Gamer, U., 1985. "Elastic-plastic deformation of a centrally heated disk". *Journal of Thermal Stresses,* ***8***, pp. 41–51.

[11] Bengeri, M., and Mack, W., 1994. "The influence of the temperature dependence of the yeld stress on the stress distribution in a thermally assembled elastic-plastic shrrink fit". *Acta Mechanica,* ***103***, pp. 243–257.

[12] Ishlinskiy, A., and Ivlev, D., 2001. *Mathematical Teory of Plasticity [in Russian]*. Fizmatlit, Moscow, Russia.

[13] Yu, M.-H., 2006. *Generalized Plasticity*. Springer Science & Business Media, Berlin.

[14] Burenin, A. A., Dats, E. P., and Murashkin, E. V., 2014. "Formation of the residual stress field under local thermal actions". *Mechanics of Solids,* ***49***(2), pp. 124–131.

[15] Dats, E., Mokrin, S., and Murashkin, E., 2015. "Calculation of the residual stresses of hollow cylinder under unsteady thermal action". *Lecture Notes in Engineering and Computer Science,* ***2018***(1), pp. 1043–1046.

[16] Dats, E., Mokrin, S., and Murashkin, E., 2016. "Calculation of the residual stress field of the thin circular plate under unsteady thermal action". *Key Engineering Materials,* ***685***, pp. 37–41.

[17] Crisfeld, M. A., 1997. *Non-linear Finite Analysis of Solids and Structures*. John Wiley & Sons, Inc., NY.

[18] Zienkiewicz, O. C., and Taylor, R. L., 2000. *The Finite Element Method*. Oxford Press, Oxford.

[19] Krabbenhoft, K., 2002. *Basic computational plasticity*. Technical University of Denmark, Lyngby, Denmark.

[20] Dats, E., Murashkin, E., and Stadnik, N., 2016. "On a multi-physics modelling framework for thermo-elastic-plastic materials processing". *Procedia Manufacturing,* ***7***, pp. 427–434.

[21] Dats, E., Murashkin, E., and Stadnik, N., 2017. "On heating of thin circular elastic-plastic plate with the yield stress depending on temperature". *Procedia Engineering,* ***173***, pp. 891–896.

[22] Boley, B., and Weiner, J., 1960. *Theory of Thermal Stresses*. Wiley, NY.

[23] von Mises, R., 1913. "Mechanik der festen korper im plastisch deformablen zustand. gottin". *Nachr. Math. Phys.,* ***1***, pp. 582–592.

[24] Hill, R., *The mathematical theory of plasticity*. Oxford University Press, Oxford, 1998.

[25] Huber, M., *Specific work of strain as a measure of material effort*. Towarzystwo Politechniczne, Czas. Techniczne, Lwow, 1903.