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AXIOMATIZABILITY OF FREE S-POSETS

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Abstract. In this work, we investigate the partially ordered monoids S over which the class of free (over a poset) S-posets is axiomatizable. Similar questions for S-sets were considered in papers of V. Gould, S. Bulman-Fleming, and A. A. Stepanova.

The questions of axiomatizability of S-sets were considered in [1, 6, 7, 14]. In [7], V. Gould obtained the description of monoids S with axiomatizable class of free S-sets. The structure of free (over a set) S-posets is similar to the structure of free S-sets, namely, free S-posets are isomorphic to the coproduct of free cyclic S-posets. Thus, the model-theoretic properties of free S-sets are easily transferred in the case of free S-posets. In particular, as we note in our work, the result of V. Gould about the axiomatizable class of free S-sets also occurs for the class of free S-posets.

In [10], the concept of an S-poset free over a poset was introduced and the structure of partially ordered monoids S with a finite number of right ideals and axiomatizable class of S-posets free over a poset were investigated. The main result of our work is a complete description of partially ordered monoids S with axiomatizable class of S-posets that are free over a poset.

The authorship of results of the given work is indivisible.

1. Some Information from Model Theory of S-Sets

Let us recall some definitions and facts from the theory of S-sets.

Let S be a monoid with identity 1. The set of the idempotents from S is denoted by E. A structure $\langle A; L_S\rangle$ of the language $L_S = \{s \mid s \in S\}$ is called a *left S-set* if for all $s, t \in S$ and $a \in A$ we have

(1) $s(ta) = (st)a$;
(2) $1a = a$.

A right S-set is defined dually.

A *partially ordered monoid* (*pomonoid*) is a monoid S together with a partial order $\le$ on S such that if $s, t, u \in S$ and $s \le t$, then $us \le ut$ and $su \le tu$. Throughout this paper, S will denote a monoid or pomonoid, which will be clear from context or specially agreed upon. Let S be a pomonoid. A structure $\langle A; L_S^{\le}\rangle$ of the language $L_S^{\le} = \{s \mid s \in S\} \cup \{\le\}$ is called a *left S-poset* if for all $s, t \in S$ and $a, a' \in A$ we have

(1) $(st)a = s(ta)$;
(2) $1a = a$;
(3) if $a \le a'$, then $sa \le sa'$;
(4) if $s \le t$, then $sa \le ta$.

In this work, we will often use the term S-(po)set to mean a left S-(po)set. We will denote an S-set $\langle A; L_S\rangle$ and S-poset $\langle A; L_S^{\le}\rangle$ as ${}_SA$ noting each time whether it is an S-set or an S-poset.

A *homomorphism* of S-posets is an order-preserving homomorphism of the corresponding S-sets. A substructure of an S-(po)set ${}_SA$ is called an *S-sub(po)set* of ${}_SA$. A *finitely generated S-sub(po)set* of an S-(po)set ${}_SA$ is an S-(po)set of the form $\bigcup\limits_{i=1}^{n} {}_SSa_i$ for some $a_1, \dots, a_n \in A$. A *cyclic S-sub(po)set* of an S-(po)set ${}_SA$ is an S-(po)set of the form ${}_SSa$ for some $a \in S$. A coproduct of S-(po)sets ${}_SA_i$ $(i \in I)$

is their disjoint union denoted $\coprod\limits_{i\in I} {}_SA_i$. The elements x and y of an S-(po)set ${}_SA$ are called *connected* (denoted $x \sim y$) if there exist $n \in \omega$, $a_0, \ldots, a_n \in A$, and $s_1, \ldots, s_n \in S$ such that $x = a_0$, $y = a_n$, and $a_i = s_i a_{i-1}$ or $a_{i-1} = s_i a_i$ for any i, $1 \leq i \leq n$. An S-sub(po)set ${}_SB$ of an S-(sub)set ${}_SA$ is called *connected* if we have $x \sim y$ for any $x, y \in B$. It is easy to check that $\sim$ is a congruence relation on an S-(po)set ${}_SA$. The classes of this relation are called *connected components* of the S-(po)set ${}_SA$.

Theorem 1.1 ([4,8])**.** *Every S-(po)set ${}_SA$ can uniquely be factorized into a coproduct of connected components.*

The concepts of free, projective, and strongly flat S-(po)sets will be important for us later on. In addition to the definitions, we will recall the algebraic characterizations of these concepts.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a functor. An object a of the category $\mathcal{A}$ is called (*left*) *free over an object b of the category $\mathcal{B}$* (by the functor $\mathcal{F}$) (see [5]) if there exists a morphism $u\colon b \to \mathcal{F}(a)$ such that for every object a' of the category $\mathcal{A}$ and every morphism $u'\colon b \to \mathcal{F}(a')$ there exists a unique morphism $v\colon a \to a'$ such that $u' = \mathcal{F}(v) \circ u$.

The category of sets as usual is denoted by SET and the category of poset by POSET. It is clear that the collection of left S-(po)sets with homomorphisms of left S-(po)sets forms a category, which is denoted by S-SET (S-POSET). Similarly the category SET-S of right S-(po)sets is defined.

Let $\mathcal{F}$ be a forgetful functor from the category S-SET to the category SET. An S-set ${}_SF$ is called *free over a set X* if ${}_SF$ as an object of the category S-SET is free over X as an object of the category SET. If in this definition we replace the category S-SET by the category S-POSET, then we obtain the definition of an S-poset *free over a set X*; if furthermore we replace the category SET by the category POSET, then we obtain the definition of an S-poset *free over a poset X*. By $\mathcal{F}r$, $\mathcal{F}r^{<}$, and $\mathcal{F}r^{\ll}$ we denote the class of S-sets that are free over a set, the class of S-posets that are free over a set, and the class of S-posets that are free over a poset, respectively.

Theorem 1.2 ([8,12])**.** *An S-(po)set ${}_SF$ is free over a set X if and only if ${}_SF \cong \coprod\limits_{x\in X} {}_SSx$, where ${}_SSx \cong {}_SS$ for all $x \in X$.*

Theorem 1.3 ([10])**.** *An S-poset ${}_SF$ is free over a poset X if and only if ${}_SF \cong \coprod\limits_{x\in X} {}_SSx$, where ${}_SSx$ is the copy of the S-poset ${}_SS$ and for all $s, t \in S$ and $x, y \in X$*

$$s_x \leq t_y \iff s \leq t \quad \text{and} \quad x \leq y, \tag{1}$$

where s_x and t_y are the copies of the elements $s, t \in S$ in Sx and Sy, respectively.

An S-(po)set ${}_SA$ is said to be *strongly flat* if the functor $-\otimes {}_SA$ from the category SET-S (POSET-S) into the category SET (POSET) preserves equalizers and pullbacks. By $\mathcal{SF}$ ($\mathcal{SF}^{<}$) we denote the class of strongly flat S-(po)sets.

Theorem 1.4 ([13])**.** *An S-set ${}_SA$ is strongly flat if and only if ${}_SA$ satisfies the conditions* (P) *and* (E):

(P) *if $sx = ty$ for $x, y \in A$ and $s, t \in S$, then there exist $z \in A$ and $s', t' \in S$ such that $x = s'z$, $y = t'z$, and $ss' = tt'$;*

(E) *if $sx = tx$ for $x \in A$ and $s, t \in S$, then there exist $z \in A$ and $s' \in S$ such that $x = s'z$ and $ss' = ts'$.*

A similar result is also true for S-posets.

Theorem 1.5 ([11])**.** *An S-poset ${}_SA$ is strongly flat if and only if ${}_SA$ satisfies the conditions* ($\mathrm{P}^{<}$) *and* ($\mathrm{E}^{<}$):

($\mathrm{P}^{<}$) *if $sx \leq ty$ for $x, y \in A$ and $s, t \in S$, then there exist $z \in A$ and $s', t' \in S$ such that $x = s'z$, $y = t'z$, and $ss' \leq tt'$;*

($\mathrm{E}^{<}$) *if $sx \leq tx$ for $x \in A$ and $s, t \in S$, then there exist $z \in A$ and $s' \in S$ such that $x = s'z$ and $ss' \leq ts'$.*

The following proposition establishes a connection between the conditions (E) and ($\mathrm{E}^<$), and it will be useful for us in the future.

Proposition 1.6 ([10]). *If an S-poset ${}_SA$ satisfies the condition* ($\mathrm{E}^<$)*, then ${}_SA$ satisfies the condition* (E).

An S-(po)set ${}_SP$ is called *projective* if for any epimorphism $\pi\colon {}_SA \to {}_SB$ and any homomorphism $\varphi\colon {}_SP \to {}_SB$ there exists a homomorphism $\psi\colon {}_SP \to {}_SA$ such that $\varphi = \pi\psi$. By $\mathcal{P}$ ($\mathcal{P}^<$) we denote the class of projective S-(po)sets. The following theorem gives us a condition that is equivalent to the projectivity of an S-(po)set.

Theorem 1.7 ([9,12]). *An S-(po)set ${}_SP$ is projective if and only if ${}_SP$ is isomorphic to a coproduct of S-(po)sets ${}_SSe$ ($e \in E$).*

The concepts of a strongly flat S-(po)set and a projective S-(po)set are associated with the concept of a perfect (po)monoid.

An S-(po)set ${}_SB$ is called a *cover* of an S-(po)set ${}_SA$ if there exists an epimorphism $f\colon {}_SB \to {}_SA$ such that the restriction of f on any proper S-sub(po)set of ${}_SB$ is not an epimorphism. If ${}_SB$ is, in addition, projective, then ${}_SB$ is a *projective cover* for ${}_SA$. A (po)monoid S is *left perfect* if every S-(po)set ${}_SA$ has a projective cover.

Theorem 1.8 ([4,10]). *For a (po)monoid S the following conditions are equivalent*:

(1) *S is left perfect (po)monoid*;
(2) $\mathcal{SF} = \mathcal{P}$ ($\mathcal{SF}^< = \mathcal{P}^<$).

The next theorem will be useful to us in the future.

Theorem 1.9 ([7]). *If S is a left perfect monoid, $St_1 \subseteq St_0$, and the S-sets ${}_SSt_1$ and ${}_SSt_0$ are isomorphic, then $St_0 = St_1$.*

Let us recall some concepts and facts from model theory and from the model theory of S-sets. Let L be a first-order language and $\mathcal{K}$ be a class of L-structures. A class $\mathcal{K}$ is called *axiomatizable* if there exists a set Z of axioms of the language L such that for any L-structure $\mathcal{A}$

$$\mathcal{A} \in \mathcal{K} \iff \mathcal{A} \vDash \Phi \quad \text{for all} \quad \Phi \in Z.$$

When we will study the axiomatizable classes below, we will frequently use the following theorem.

Theorem 1.10 ([2]). *If a class $\mathcal{K}$ is axiomatizable, then $\mathcal{K}$ is closed under the formation of ultraproducts.*

In [7,10], there were described (po)monoids with axiomatizable classes of free, projective, and strongly flat S-(po)sets. We will give here the results from these papers which will be used further.

For any $s, t \in S$ let us define the sets

$$r(s,t) = \{u \in S \mid su = tu\}, \quad R(s,t) = \{\langle u, v\rangle \in S \times S \mid su = tv\},$$
$$r^<(s,t) = \{u \in S \mid su \le tu\}, \quad R^<(s,t) = \{\langle u, v\rangle \in S \times S \mid su \le tv\}.$$

Theorem 1.11 ([7]). *The class $\mathcal{SF}$ is axiomatizable if and only if for any $s, t \in S$*

(1) *the set $r(s,t)$ is empty or finitely generated as a right ideal of S*;
(2) *the set $R(s,t)$ is empty or finitely generated as an S-subset of the right S-set $(S \times S)_S$.*

Theorem 1.12 ([7]). *The class $\mathcal{P}$ is axiomatizable if and only if the class $\mathcal{SF}$ is axiomatizable and the monoid S is left perfect.*

For the formulation of an axiomatizability criterion of the class of free S-sets we will need some new concepts. Let $e \in E$ and $s, x \in S$. We say that $s = xy$ is an *e-good factorization on x* if $y \notin wS$ for any $w \in S$ such that $e = xw$ and $Sw = Se$.

Theorem 1.13 ([7]). *The class $\mathcal{F}r$ is axiomatizable if and only if the class $\mathcal{P}$ is axiomatizable and the monoid S satisfies the following condition*:

$$\text{for any } e \in E \setminus \{1\} \text{ there exists a finite set } T \subseteq S \text{ such that any } s \in S \text{ has an } e\text{-good factorization on } x \text{ for some } x \in T. \tag{$*$}$$

From the proof of this theorem (see [7]) we have immediately the following proposition.

Corollary 1.14. *Let S be a pomonoid. If the class $\mathcal{F}r^{\ll}$ is axiomatizable, then the monoid S satisfies condition $(*)$.*

Theorem 1.15 ([10]). *If any ultrapower of an S-poset ${}_SS$ is free over a poset, then the pomonoid S is left perfect.*

Theorem 1.16 ([10]). *Let any ultrapower of an S-poset ${}_SS$ be free over a poset. Then for any $s, t \in S$*

(1) *the set $r^{<}(s,t)$ is empty or finitely generated as a right ideal of S;*
(2) *the set $R(s,t)$ is empty or finitely generated as an S-subset of the right S-set $(S \times S)_S$.*

We say that a monoid S has the *condition of finite right solutions* if

$$\forall\, s \in S\ \exists\, n_s \in \mathbb{N}\ \forall\, t \in S\ |\{x \in S \mid sx = t\}| \leq n_s.$$

Proposition 1.17 ([7]). *Let S be a monoid. If any ultrapower of the S-set ${}_SS$ is projective, then S satisfies the condition of finite right solutions.*

2. Preliminary Results

In this section, we give the lemmas that will be used for the proof of our crucial result. Some of these lemmas are of interest in themselves.

Lemma 2.1. *Let S be a left perfect pomonoid. Then S is a left perfect monoid.*

Proof. Let S be a left perfect pomonoid. By Theorem 1.8, it is enough to prove that $\mathcal{SF} = \mathcal{P}$. Let ${}_SA$ be a strongly flat S-set. We define a relation $\leq$ on A as follows:

$$sa \leq tb \iff \exists\, u \in A\ \exists\, s_1, s_2, t_1, t_2 \in S\colon a = s_1u,\ b = t_1u,\ ss_1u = s_2u,\ tt_1u = t_2u,\ s_2 \leq t_2, \tag{2}$$

where $a, b \in A$, $s, t \in S$. We claim that $\leq$ is a partial order on A. Clearly, $\leq$ is a reflexive relation.

We will show the transitivity of the relation $\leq$. Let $a, b, c \in A$ and $s, t, r \in S$ satisfy $sa \leq tb \leq rc$. Then there exist $u, v \in A$, $s_1, s_2, t_1, t_2, t', t'', r_1, r_2 \in S$ such that condition (2) holds and $b = t'v$, $c = r_1v$, $tt'v = t''v$, $rr_1v = r_2v$, and $t'' \leq r_2$. Note that $t_2u = tt_1u = tb = tt'v = t''v$. Since ${}_SA$ is strongly flat, by Theorem 1.4 the S-set ${}_SA$ satisfies condition (P). Hence the equality $t_2u = t''v$ implies that $u = s_3w$, $v = r_3w$, and $t_2s_3 = t''r_3$ for some $w \in A$ and $s_3, r_3 \in S$. Then $a = s_1s_3w$, $c = r_1r_3w$, $ss_1s_3w = ss_1u = s_2u = s_2s_3w$, $rr_1r_3w = rr_1v = r_2v = r_2r_3w$, and $s_2s_3 \leq t_2s_3 = t''r_3 \leq r_2r_3$. Therefore, $\leq$ is a transitive relation.

To show the symmetry of the relation $\leq$, suppose now that $sa \leq tb \leq sa$ for $a, b \in A$ and $s, t \in S$. Thus, there exist $u, v \in A$, $s_1, s_2, t_1, t_2, t', t'', r_1, r_2 \in S$ such that condition (2) holds and $b = t'v$, $a = r_1v$, $tt'v = t''v$, $sr_1v = r_2v$, and $t'' \leq r_2$. By condition (P), from the equality $t_2u = t''v$, which is proved as above, there follows the existence of $w \in A$ and $s_3, r_3 \in S$ such that $u = s_3w$, $v = r_3w$, and $t_2s_3 = t''r_3$. Note that $s_2s_3w = s_2u = ss_1u = sa = sr_1v = r_2v = r_2r_3w$. As $s_2s_3w = r_2r_3w$ and condition (E) holds, there exist $x \in A$ and $t \in S$ such that $w = tx$ and $s_2s_3t = r_2r_3t$. Since $s_2s_3t \leq t_2s_3t = t''r_3t \leq r_2r_3t$, we have $s_2s_3t = t_2s_3t$ and $sa = s_2s_3w = s_2s_3tx = t_2s_3tx = t_2s_3w = t_2u = tt_1u = tb$. Therefore, $\leq$ is a symmetric relation.

It is easy to check that for any $s, t, u, v \in S$ and $a, b \in A$ if $u \leq v$ and $sa \leq tb$, then $usa \leq vtb$. Thus, ${}_SA$ is an S-poset.

Let us show that the S-poset ${}_SA$ satisfies condition (E$^{<}$). Suppose that $sa \leq ta$ for some $s, t \in S$ and $a \in A$. Then there exist $u \in A$, $s_1, s_2, t_1, t_2 \in S$ such that $a = s_1u = t_1u$, $ss_1u = s_2u$, $tt_1u = t_2u$, and

$s_2 \leq t_2$. Since $ss_1u = s_2u$ and the S-set ${}_SA$ satisfies condition (E), there exist $u_1 \in A$ and $r_1 \in S$ such that $u = r_1u_1$ and $ss_1r_1 = s_2r_1$. As $tt_1r_1u_1 = tt_1u = t_2u = t_2r_1u_1$, i.e., $tt_1r_1u_1 = t_2r_1u_1$, and the S-set ${}_SA$ satisfies condition (E), we have that there exist $u_2 \in A$ and $r_2 \in S$ such that $u_1 = r_2u_2$ and $tt_1r_1r_2 = t_2r_1r_2$. Since $s_1r_1r_2u_2 = s_1r_1u_1 = s_1u = t_1u = t_1r_1u = t_1r_1r_2u_2 = t_1r_1r_2u_2$, i.e., $s_1r_1r_2u_2 = t_1r_1r_2u_2$, and the S-set ${}_SA$ satisfies condition (E), we have that there exist $u_3 \in A$ and $r_3 \in S$ such that $u_2 = r_3u_3$ and $t_1r_1r_2r_3 = s_1r_1r_2r_3$. Hence $a = s_1r_1r_2r_3u_3$ and $ss_1r_1r_2r_3 = s_2r_1r_2r_3 \leq t_2r_1r_2r_3 = tt_1r_1r_2r_3 = ts_1r_1r_2r_3$. Thus, ${}_SA$ satisfies condition ($\mathrm{E}^<$).

We claim that ${}_SA$ satisfies condition ($\mathrm{P}^<$). Let $sa \leq tb$. Then condition (2) holds. The equality $ss_1u = s_2u$ together with condition (E) implies the existence of $u_1 \in A$ and $r_1 \in S$ such that $u = r_1u_1$ and $s_2r_1 = ss_1r_1$. Since $t_2r_1u_1 = tt_1r_1u_1$ and the S-set ${}_SA$ satisfies condition (E), there exist $u_2 \in A$ and $r_2 \in S$ such that $u_1 = r_2u_2$ and $t_2r_1r_2 = tt_1r_1r_2$. Thus, $a = s_1r_1r_2u_2$, $b = t_1r_1r_2u_2$, and $ss_1r_1r_2 = s_2r_1r_2 \leq t_2r_1r_2 = tt_1r_1r_2$.

By Theorem 1.5, the S-poset ${}_SA$ is strongly flat. As S is a left perfect pomonoid then by Theorem 1.8 the S-poset ${}_SA$ is projective. By Theorem 1.7, we deduce that the S-set ${}_SA$ is isomorphic to a coproduct of the cyclic S-sets generated by idempotents, i.e., ${}_SA$ is a projective S-set. $\square$

Let S be a monoid. We will define an equivalence relation $\mathcal{H}$ (see [3]) on S as follows:

$$s\mathcal{H}t \iff Ss = St \text{ and } sS = tS,$$

where $s, t \in S$. By $\mathcal{H}_1$ we denote the $\mathcal{H}$-class of the element 1. Note that the set $\mathcal{H}_1$ is the group of units of the monoid S.

Lemma 2.2. *If S is a left perfect monoid, $t \in S$, and $S = tS$, then $t \in \mathcal{H}_1$.*

Proof. Let $t \in S$ and $S = tS$. Then there exists $t' \in S$ such that $tt' = 1$. Note that the mapping $\varphi\colon {}_SS \to {}_SSt$ defined by $\varphi(s) = st$ for any $s \in S$ is an isomorphism of S-sets. Indeed, if $kt = lt$, then $ktt' = ltt'$, i.e., $k = l$ for any $k, l \in S$. Since $St \subseteq S$, we have by Theorem 1.9 that $St = S$, i.e., $t \in \mathcal{H}_1$. $\square$

Lemma 2.3. *If there are $s, t \in \mathcal{H}_1$ such that $s < t$, then there is an ascending chain in a pomonoid S.*

Proof. Assume that $s, t \in \mathcal{H}_1$ and $s < t$. Since $s \in \mathcal{H}_1$, there exists an element $s^{-1} \in S$ such that s^{-1} is the inverse of s. Let us multiply the inequality $s < t$ by s^{-1} from the right. Then $1 \leq ts^{-1}$. If $1 = ts^{-1}$, then $s = ts^{-1}s = t$, a contradiction. Hence $1 < ts^{-1}$. Denote ts^{-1} by r. Then $1 < r$. Let us multiply this inequality by r^i ($i \in \omega$). We have $r^i \leq r^{i+1}$. Since $\mathcal{H}_1$ is a group, we see that $r^i \in \mathcal{H}_1$ for any $i \in \omega$. If $r^i = r^{i+1}$ for some $i \in \omega$, then $1 = r^i(r^i)^{-1} = r^{i+1}(r^i)^{-1} = r$, that is not so. Thus, we obtain the ascending chain $1 < r < r^2 < r^3 < \dots$. $\square$

Lemma 2.4. *Let S be a pomonoid. If for any $s, t \in S$ the set $r^<(s,t)$ is either empty or finitely generated as a right ideal of S, and the set $R(s,t)$ is either empty or finitely generated as an S-subset of the right S-set $(S \times S)_S$, then for any $s, t \in S$ the set $r(s,t)$ is either empty or finitely generated as a right ideal of S.*

Proof. Let $s, t \in S$ and $r(s,t) \neq \varnothing$. Note that $r(s,t) \subseteq r^<(s,t)$ and $r(s,t) \subseteq r^<(t,s)$. By assumption,

$$r^<(s,t) = \bigcup_{x \in X} xS, \quad r^<(t,s) = \bigcup_{y \in Y} yS$$

for some finite sets $X \subseteq S$ and $Y \subseteq S$, in particular, $sx \leq tx$ and $ty \leq sy$ for any $x \in X$ and $y \in Y$. Furthermore, for any $x, y \in S$ we have

$$R(x,y) = \bigcup_{\langle u,v \rangle \in W_{xy}} \langle u,v \rangle S$$

for some finite set $W_{xy} \in S \times S$, in particular, $xu = yv$ for any $\langle u,v \rangle \subseteq W_{xy}$. For $x \in X$ by U_x we denote a set

$$\{u \in S \mid \langle u,v \rangle \in W_{xy} \text{ for some } y \in Y \text{ and } v \in S\}.$$

Let us prove the equality

$$r(s,t) = \bigcup_{x\in X} \bigcup_{u\in U_x} xuS.$$

Suppose that $w \in r(s,t)$. From $r(s,t) \subseteq r^<(s,t)$ and $r(s,t) \subseteq r^<(t,s)$ it follows that $w = xw' = yw''$ for some $x \in X$, $y \in Y$, $w', w'' \in S$ and $\langle w', w''\rangle \in R(x,y)$. Hence $\langle w', w''\rangle = \langle u, v\rangle z$ for some $\langle u, v\rangle \in W_{xy}$ and $z \in S$. Then $w = xuz$ and $w \in \bigcup\limits_{x\in X} \bigcup\limits_{u\in U_x} xuS$. Thus, the inclusion

$$r(s,t) \subseteq \bigcup_{x\in X} \bigcup_{u\in U_x} xuS$$

is proved.

Let $x \in X$, $u \in U_x$, and $w \in S$. Then $xu = yv$ for some $y \in Y$ and $v \in S$. Hence $sxu = syv$ and $txu = tyv$. From $sx \leq tx$ and $ty \leq sy$ it follows that $sxu \leq txu = tyv \leq syv = sxu$, i.e., $sxu = txu$. Thus, $sxuw = txuw$ and $xuw \in r(s,t)$. Thus, the inclusion

$$\bigcup_{x\in X} \bigcup_{u\in U_x} xuS \subseteq r(s,t)$$

is proved. □

Lemma 2.5. *Let S be pomonoid. If the class $\mathcal{F}r^{\ll}$ is axiomatizable, then the class $\mathcal{F}r$ is axiomatizable.*

Proof. Let the class $\mathcal{F}r^{\ll}$ be axiomatizable. By Corollary 1.14, the monoid S satisfies the condition $(*)$. By Theorem 1.10, any ultrapower of the S-poset ${}_SS$ is a free S-poset over a poset. By Theorem 1.16, for any $s,t \in S$ the set $r^<(s,t)$ is either empty or finitely generated as a right ideal of S and the set $R(s,t)$ is either empty or finitely generated as an S-subset of the right S-set $(S \times S)_S$. By Lemma 2.4, for any $s,t \in S$ the set $r(s,t)$ is either empty or finitely generated as a right ideal of S. By Theorem 1.11, the class $\mathcal{SF}$ is axiomatizable. By Theorem 1.15, the pomonoid S is left perfect. Hence by Lemma 2.1 the monoid S is left perfect too. Thus, by Theorem 1.12 the class $\mathcal{P}$ is axiomatizable. Thus, by Theorem 1.13 the class $\mathcal{F}r$ is axiomatizable. □

3. Axiomatizability of the Class of Free S-Posets

The following theorem characterizes pomonoids S such that the class of S-posets that are free over a set is axiomatizable. The proof of this theorem is analogous to the proof of Theorem 1.13 and so we do not give it here.

Theorem 3.1. *The class $\mathcal{F}r^<$ is axiomatizable if and only if the class $\mathcal{P}^<$ is axiomatizable and S satisfies the following condition*:

$$\text{for any } e \in E \setminus \{1\} \text{ there exists a finite set } T \subseteq S \text{ such that any } s \in S \text{ has an } e\text{-good factorization on } x \text{ for some } x \in T. \qquad (*)$$

The crucial result of this work is Theorem 3.2, which describes pomonoids S with axiomatizable class of S-posets that are free over a poset. To formulate the following theorem we need some notations.

Let S be a pomonoid and $s,t \in S$, $r \in \mathcal{H}_1$. Let us define the following sets:

$\langle x,y\rangle \in L_1(s,t) \iff x$ is the maximal element of a poset S such that $sx \leq ty$;

$\langle x,y\rangle \in L_2(s,t) \iff x$ is the maximal element of a poset S such that $sx < ty$ and either $sx \notin tS$ or $ty \notin sS$;

$\langle x,y\rangle \in L_3(r) \iff y \neq ry$ and x is the maximal element of a poset S such that $x \leq ry$ and $x \leq y$.

Theorem 3.2. *Let S be a pomonoid. Then the class $\mathcal{F}r^{\ll}$ is axiomatizable if and only if*

(1) *the class $\mathcal{F}r$ is axiomatizable*;
(2) *there are no ascending or descending chains in the poset S*;
(3) *for any $\rho \in S \times S$ the set $r^<(\rho)$ is either empty or finitely generated as a right ideal of S*;

(4) *for any* $i \in \{1,2\}$ *and* $\rho \in S \times S$ *either the set* $L_i(\rho)$ *is empty or there is a finite set* $L^i_\rho \subseteq L_i(\rho)$ *such that* $L_i(\rho) \subseteq \bigcup\limits_{\langle x,y\rangle \in L^i_\rho} \langle x, y\rangle S$;

(5) *for any* $s \in \mathcal{H}_1$ *either the set* $L_3(s)$ *is empty or there is a finite set* $L^3_s \subseteq L_3(s)$ *such that* $L_3(s) \subseteq \bigcup\limits_{\langle x,y\rangle \in L^3_s} \langle x, y\rangle S$.

Proof. Necessity. Let the class $\mathcal{F}r^{\ll}$ be axiomatizable. From Lemma 2.5 there follows (1).

Let us prove (2). Assume that there exists an ascending chain $a_0 < a_1 < a_2 < \dots < a_n < \dots$ in the poset S. Let D be a nonprincipal ultrafilter on ω. By Theorem 1.10, ${}_SS^\omega/D \in \mathcal{F}r^{\ll}$.

We claim that $S \cdot \bar{1}/D$ is a connected component of the S-(po)set ${}_SS^\omega/D$, where $\bar{1}(j) = 1$ $(j \in \omega)$. Let $\bar{1}/D = t\bar{c}/D$ for some $t \in S$ and $\bar{c} \in S^\omega$. Since a free S-set is projective, by Proposition 1.17 the set $\{x \in S \mid tx = 1\}$ is finite. Hence $\bar{c}/D \in S \cdot \bar{1}/D$.

Consider $\bar{a}, \bar{a}_i \in S^\omega$, where $\bar{a}(j) = a_j$ and $\bar{a}_i(j) = a_i$ $(i, j \in \omega)$. It is clear that $\bar{a}_i/D < \bar{a}/D$, $\bar{a}_i/D \in S \cdot \bar{1}/D$, and $\bar{a}/D \notin S \cdot \bar{1}/D$. Since ${}_SS^\omega/D \in \mathcal{F}r^{\ll}$, we have that there exists an isomorphism of the connected component of the S-poset ${}_SS^\omega/D$, which contains the element $\bar{a}/D$, into the connected component ${}_SS \cdot \bar{1}/D$. Let $\bar{b}/D$ be the image of the element $\bar{a}/D$ under this isomorphism and $\bar{b}(j) = b \in S$ $(j \in \omega)$. Since $\bar{a}_i/D < \bar{a}/D$ $(i \in \omega)$, by Theorem 1.3 we have that $\bar{b}/D < \bar{a}/D$ and $\bar{a}_i/D < \bar{b}/D$ for any $i \in \omega$. Consequently, there exists $j \in \omega$ such that $b < a_j$ and $a_i < b$ for any $i \in \omega$, i.e., $a_i < a_j$ for any $i \in \omega$, a contradiction. In the same way, it is proved that there are no descending chains in the poset S.

From Theorem 1.16 there follows (3).

Let us prove (4). Assume that $i \in \{1, 2\}$ and there exists $\rho(s, t) \in S \times S$ such that condition (4) does not hold. Let

$$\{\langle x_\alpha, y_\alpha\rangle \in L_i(\rho) \mid \alpha < \gamma\}$$

be a set of minimum cardinality γ such that $L_i(\rho) \subseteq \bigcup\limits_{\alpha<\gamma} \langle x_\alpha, y_\alpha\rangle S$. Since γ is infinite, it must be a limit ordinal. We can assume that

$$\langle x_\beta, y_\beta\rangle \notin \bigcup_{\alpha<\beta} \langle x_\alpha, y_\alpha\rangle S \tag{3}$$

for any $\beta < \gamma$. Let D be a nonprincipal ultrafilter on γ. As the class $\mathcal{F}r^{\ll}$ is axiomatizable, we have ${}_SS^\gamma/D \in \mathcal{F}r^{\ll}$. Let $\bar{x}, \bar{y} \in S^\gamma$ such that $\bar{x}(\alpha) = x_\alpha$, $\bar{y}(\alpha) = y_\alpha$ $(\alpha \in \gamma)$. Note that $s\bar{x}/D \le t\bar{y}/D$ and for $i = 2$ either $s\bar{x}/D \notin tS^\gamma/D$ or $t\bar{y}/D \notin sS^\gamma/D$.

Suppose that the elements $\bar{x}/D$ and $\bar{y}/D$ are in different connected components of the S-poset ${}_SS^\gamma/D$. Since ${}_SS^\gamma/D \in \mathcal{F}r^{\ll}$, there exists an isomorphism of the connected component of the S-poset ${}_SS^\gamma/D$, which contains the element $\bar{x}/D$, into the connected component of the S-poset ${}_SS^\gamma/D$, which contains the element $\bar{y}/D$. Let $\bar{x}'/D$ be the image of the element $\bar{x}/D$ under this isomorphism, $\bar{x}'(\alpha) = x'_\alpha$ for any $\alpha \in \gamma$. Thus, for $i = 2$ either $s\bar{x}'/D \notin tS^\gamma/D$ or $t\bar{y}/D \notin sS^\gamma/D$. By Theorem 1.3, $\bar{x}/D < \bar{x}'/D$ and $s\bar{x}'/D \le t\bar{y}/D$. Then $s\bar{x}/D < s\bar{x}'/D \le t\bar{y}/D$. Hence there exists $\alpha \in \gamma$ such that $x_\alpha < x'_\alpha$, $sx_\alpha < sx'_\alpha \le ty_\alpha$ and for $i = 2$ either $sx'_\alpha \notin tS$ or $ty_\alpha \notin sS$, contradicting the condition $\langle x_\alpha, y_\alpha\rangle \in L_i(\rho)$.

Let the elements $\bar{x}/D$ and $\bar{y}/D$ be in the same connected component of the S-poset ${}_SS^\gamma/D$. By Theorem 1.3, there exists an isomorphism of this connected component into the S-poset ${}_SS$. Let $\bar{h}/D$ be the inverse image of 1, $\bar{x}/D$ be the inverse image of $k \in S$, and $\bar{y}/D$ be the inverse image of $l \in S$ under this isomorphism. From the inequality $s\bar{x}/D \le t\bar{y}/D$ it follows that $sk \le tl$ and for $i = 2$ either $sk \notin tS$ or $tl \notin sS$. We will show that $\langle k, l\rangle \in L_i(\rho)$. Suppose $k \le k'$, $sk \le sk' \le tl$ and for $i = 2$ either $sk' \notin tS$ or $tl \notin sS$. Let us multiply these inequalities from the right by $\bar{h}/D$ and denote $k'\bar{h}/D$ by $\bar{x}'/D$. Then $\bar{x}/D \le \bar{x}'/D$, $s\bar{x}/D \le s\bar{x}'/D \le t\bar{y}/D$ and for $i = 2$ either $s\bar{x}'/D \notin tS^\gamma/D$ or $t\bar{y}/D \notin sS^\gamma/D$. Hence

$$I = \{\alpha \in \gamma \mid x_\alpha \le x'_\alpha,\ sx_\alpha \le sx'_\alpha \le ty_\alpha \text{ and for } i = 2 \text{ either } sx'_\alpha \notin tS \text{ or } ty_\alpha \notin sS\} \in D.$$

Since $\langle x_\alpha, y_\alpha\rangle \in L_i(\rho)$ for any $\alpha \in \gamma$, we have $I \subseteq \{\alpha < \gamma \mid x_\alpha = x'_\alpha\}$. Consequently, $\{\alpha \in \gamma \mid x_\alpha = x'_\alpha\} \in D$ and $\bar{x}/D = \bar{x}'/D$, whence $k = k'$ and $\langle k, l\rangle \in L_i(\rho) \subseteq \bigcup\limits_{\alpha\in\gamma} \langle x_\alpha, y_\alpha\rangle S$, i.e., $\langle k, l\rangle = \langle x_\alpha, y_\alpha\rangle r$

for some $\alpha \in \gamma$ and $r \in S$. On the other hand, $\langle \bar{x}/D, \bar{y}/D \rangle = \langle k, l \rangle \bar{h}/D$. Then there exists $\beta > \alpha$ such that $\langle x_\beta, y_\beta \rangle \in \langle k, l \rangle S \subseteq \langle x_\alpha, y_\alpha \rangle S$, contradicting (3).

Let us prove (5). Suppose that there exists $s \in \mathcal{H}_1$ such that (5) is not true. As in the proof of (4) for a set $L_3(s)$ we construct the set

$$\{\langle x_\alpha, y_\alpha \rangle \in L_3(s) \mid \alpha \in \gamma\}$$

such that (3) holds for all $\beta < \gamma$, D is the ultrafilter on γ, and the elements $\bar{x}/D$ and $\bar{y}/D$ belong to S^γ/D. Clearly, $\bar{x}/D \le \bar{y}/D$, $\bar{x}/D \le s\bar{y}/D$, and $\bar{y}/D \ne s\bar{y}/D$.

Now suppose that the elements $\bar{x}/D$ and $\bar{y}/D$ are in different connected components of the S-poset ${}_SS^\gamma/D$. Let $\bar{h}/D$ be a generating element of the connected component of the S-poset ${}_SS^\gamma/D$ that contains $\bar{x}/D$ and $\bar{h}'/D$ be a generating element of the connected component of the S-poset ${}_SS^\gamma/D$ that contains $\bar{y}/D$; $\bar{h}'(\alpha) = h'_\alpha$ for all $\alpha \in \gamma$. There is an isomorphism of the S-poset ${}_SS\bar{h}/D$ into the S-poset ${}_SS\bar{h}'/D$. We can assume that $\bar{h}'/D$ is the image of the element $\bar{h}/D$ under this isomorphism. By Theorem 1.3, $\bar{h}/D < \bar{h}'/D$, $t\bar{h}'/D \le r\bar{h}'/D = \bar{y}/D$ and $t\bar{h}'/D \le sr\bar{h}'/D = s\bar{y}/D$. Thus, $\bar{x}/D < t\bar{h}'/D \le \bar{y}/D$ and $\bar{x}/D < t\bar{h}'/D \le s\bar{y}/D$. Hence there exists $\alpha \in \gamma$ such that $x_\alpha < th'_\alpha$, $th'_\alpha \le y_\alpha$, and $th'_\alpha \le sy_\alpha$, contradicting the condition $\langle x_\alpha, y_\alpha \rangle \in L_3(s)$.

Suppose that the elements $\bar{x}/D$ and $\bar{y}/D$ are in the same connected component of the S-poset ${}_SS^\gamma/D$. By Theorem 1.3, there exists an isomorphism of this connected component into the S-poset ${}_SS$. Let $\bar{h}/D$ be the inverse image of 1, $\bar{x}/D$ be the inverse image of k, and $\bar{y}/D$ be the inverse image of l under this isomorphism. Since $\bar{x}/D \le \bar{y}/D$, $\bar{x}/D \le s\bar{y}/D$, and $\bar{y}/D \ne s\bar{y}/D$, we see that $k \le l$, $k \le sl$, and $l \ne sl$. We will show that $\langle k, l \rangle \in L_3(s)$. Let $k \le k'$, $k' \le l$, and $k' \le sl$. Let us multiply these inequalities from the right by $\bar{h}/D$. Then $\bar{x}'/D \le \bar{y}/D$ and $\bar{x}'/D \le s\bar{y}/D$, where $\bar{x}'/D = k'\bar{h}/D$. Hence $I = \{\alpha \in \gamma \mid x_\alpha \le x'_\alpha,\ x'_\alpha \le y_\alpha \text{ and } x'_\alpha \le sy_\alpha\} \in D$. Since $\langle x_\alpha, y_\alpha \rangle \in L_3(s)$ for any $\alpha \in \gamma$ we have $I \subseteq \{\alpha < \gamma \mid x_\alpha = x'_\alpha\}$. Consequently, $\{\alpha \in \gamma \mid x_\alpha = x'_\alpha\} \in D$ and $\bar{x}/D = \bar{x}'/D$, whence $k = k'$ and $\langle k, l \rangle \in L_3(s) \subseteq \bigcup\limits_{\alpha \in \gamma} \langle x_\alpha, y_\alpha \rangle S$, i.e., $\langle k, l \rangle = \langle x_\alpha, y_\alpha \rangle r$ for some $\alpha \in \gamma$ and $r \in S$. On the other hand, $\langle \bar{x}/D, \bar{y}/D \rangle = \langle k, l \rangle \bar{h}/D$. We deduce that there exists $\beta > \alpha$ such that $\langle x_\alpha, y_\alpha \rangle \in \langle k, l \rangle S \subseteq \langle x_\beta, y_\beta \rangle S$, contradicting (3).

Sufficiency. Suppose that conditions (1)–(5) of the theorem hold. Let $\rho = (s, t) \in S \times S$. If $r^<(\rho) \ne \varnothing$, then we choose and fix a finite set $\bar{r}_\rho$ of generators of $r^<(\rho)$. We define a sentence $\Phi_r(\rho)$ of $L_S^\le$ as follows: if $r^<(\rho) = \varnothing$, then

$$\Phi_r(\rho) \leftrightharpoons \forall x\ \neg(sx \le tx),$$

and, on the other hand, if $r^<(\rho) \ne \varnothing$, we put

$$\Phi_r(\rho) \leftrightharpoons \forall x \left(sx \le tx \to \exists z \bigvee_{u \in \bar{r}_\rho} x = uz \right).$$

Let

$$\alpha_\rho(x, y) \rightleftharpoons sx < ty \wedge \big(\neg\exists u\ (sx = tu) \vee \neg\exists u\ (ty = su)\big), \quad \gamma_s(x, y) \rightleftharpoons x \le y \wedge x \le sy \wedge y \ne sy.$$

We define a sentence $\Phi_{L_1}(\rho)$ of $L_S^\le$ as follows: if $L_1(\rho) = \varnothing$, then

$$\Phi_{L_1}(\rho) \leftrightharpoons \forall xy\ \neg(sx \le ty),$$

otherwise, if $L_1(\rho) \ne \varnothing$, we put

$$\Phi_{L_1}(\rho) \rightleftharpoons \forall xy \Bigg(sx \le ty \\ \to \exists z \Bigg(sz \le ty \wedge x \le z \wedge \forall z'\ (z \le z' \wedge sz' \le ty \to z = z') \wedge \exists w \bigvee_{\langle u,v \rangle \in L_\rho^1} \langle z, y \rangle = \langle u, v \rangle w \Bigg)\Bigg).$$

We define a sentence $\Phi_{L_2}(\rho)$ of $L_S^{\leq}$ as follows: if $L_2(\rho) = \varnothing$, then

$$\Phi_{L_2}(\rho) \leftrightharpoons \forall\, xy\ \neg\alpha_\rho(x, y),$$

otherwise, if $L_2(\rho) \neq \varnothing$, we put

$$\Phi_{L_2}(\rho) \rightleftharpoons \forall\, xy\ \Bigg(\alpha_\rho(x, y) \\ \to \exists\, z\ \Bigg(\alpha_\rho(z, y) \wedge x \leq z \wedge \forall\, z'\ (\alpha_\rho(z', y) \wedge z \leq z' \to z = z') \wedge \exists\, w \bigvee_{\langle u,v\rangle \in L^2_\rho} \langle z, y\rangle = \langle u, v\rangle w\Bigg)\Bigg).$$

For any element $s \in \mathcal{H}_1$ we define a sentence $\Phi_{L_3}(s)$ of $L_S^{\leq}$ as follows: if $L_3(s) = \varnothing$, then

$$\Phi_{L_3}(s) \leftrightharpoons \forall\, xy\ \neg\gamma_s(x, y),$$

otherwise, if $L_3(s) \neq \varnothing$, we put

$$\Phi_{L_3}(s) \rightleftharpoons \forall\, xy\ \Bigg(\gamma_s(x, y) \\ \to \exists\, z\ \Bigg(x \leq z \wedge \gamma_s(z, y) \wedge \forall\, z'\ (z \leq z' \wedge \gamma_s(z', y) \to z = z') \wedge \exists\, w \bigvee_{\langle u,v\rangle \in L^3_s} \langle z, y\rangle = \langle u, v\rangle w\Bigg)\Bigg).$$

Since the class $\mathcal{F}r$ is axiomatizable, there exists a set of axioms for this class. By $\Sigma_{\mathcal{F}r}$ we denote this set. We claim that

$$\Sigma_{\mathcal{F}r^{\ll}} = \Sigma_{\mathcal{F}r} \cup \{\Phi_r(\rho) \mid \rho \in S \times S\} \cup \{\Phi_{L_1}(\rho) \mid \rho \in S \times S\} \cup \{\Phi_{L_2} < (\rho) \mid \rho \in S \times S\} \cup \{\Phi_{L_3}(s) \mid s \in \mathcal{H}_1\}$$

axiomatizes the class $\mathcal{F}r^{\ll}$.

Suppose first that ${}_SA \models \Sigma_{\mathcal{F}r^{\ll}}$. By Theorem 1.1, ${}_SA = \coprod_{x\in X} {}_SA_x$, where ${}_SA_x$ are the connected components. Let $x \in X$. Since ${}_SA \models \Sigma_{\mathcal{F}r}$, we have that the S-set ${}_SA_x$ is isomorphic to the S-set ${}_SS$. Fix $h_x \in A_x$ and the mapping $\varphi\colon {}_SA_x \to {}_SS$ such that ${}_SA_x = {}_SSh_x$, $\varphi(h_x) = 1$, and φ is an isomorphism of S-sets. We claim that S-posets ${}_SSh_x$ and ${}_SS$ are isomorphic. It is enough to prove that

$$sh_x \leq th_x \iff s \leq t$$

for any $s, t \in S$. If $s \leq t$, then by the definition of an S-poset we have $sh_x \leq th_x$. Let $sh_x \leq th_x$. Since ${}_SA_x \vDash \Phi_r(s, t)$, there exist $u \in Sh_x$ and $r \in S$ such that $h_x = ru$ and $sr \leq tr$. Since $u \in Sh_x$, there exists $r' \in S$ such that $u = r'h_x$. Consequently, $h_x = rr'h_x$ and $\varphi(h_x) = \varphi(rr'h_x)$, i.e., $1 = rr'$. Let us multiply the inequality $sr \leq tr$ by r' from the right. We have $srr' \leq trr'$. Hence $s \leq t$. Thus, the S-posets ${}_SSh_x$ and ${}_SS$ are isomorphic.

We note that the relation $\leq$ on the poset $\mathcal{H}_1$ coincides with the relation of equality. Indeed, let $z_1 < z_2$ for some $z_1, z_2 \in \mathcal{H}_1$. Then $1 < z_2z_1^{-1}$. We denote $z_2z_1^{-1}$ by u. Thus, we have a chain $1 < u \leq u^2 \leq u^3 \leq \dots$. If $u^i = u^j$ for some $i, j \in \omega$, $j > i$, then in view of $u^i \in \mathcal{H}_1$ we have $1 = u^{j-i}$, whence $1 = u$, a contradiction. Thus, there is an ascending chain in the poset S, contradicting (2).

Wed define on the set X the relation $\leq$ in the following way:

$$x \leq y \iff \exists\, z \in \mathcal{H}_1\colon h_x \leq zh_y$$

for all $x, y \in X$. Since on the poset $\mathcal{H}_1$ the relation $\leq$ coincides with the relation of equality, we have that this relation on X is a partial order relation. We claim that ${}_SA$ is an S-poset free over the poset X. Let $h_1, h_2 \in \{h_x \mid x \in X\}$, $h_1 \neq h_2$.

Suppose that $h_1 < z_0h_2$. We will show that there exists $z \in \mathcal{H}_1$ such that $h_1 < zh_2$ and $z \leq z_0$. If $z_0 \in \mathcal{H}_1$, then we suppose that $z = z_0$. Consider $z_0 \notin \mathcal{H}_1$. Since the class $\mathcal{F}r$ is axiomatizable, by Theorem 1.13 the class $\mathcal{P}$ is axiomatizable too and by Theorem 1.12 the monoid S is left perfect. Then by Lemma 2.2 $1 \notin z_0S$. As ${}_SA \models \Phi_{L_2}(1, z_0)$, we have that there is $z_1 \in S$ such that $z_1h_2 \leq z_0h_2$, $h_1 < z_1h_2$, and $z_1 \notin z_0S$. If $z_1 \in \mathcal{H}_1$, then we suppose $z = z_1$. Otherwise by ${}_SA \models \Phi_{L_2}(1, z_1)$ we get an element $z_2 \in S$ such that $z_2 h_2 \le z_1 h_2$, $h_1 < z_2 h_2$, and $z_2 \notin z_1 S$. If $z_2 \in \mathcal{H}_1$, then we suppose $z = z_2$. Otherwise we continue this process. As a result we have either an element $z_i \in \mathcal{H}_1$ such that $h_1 < z_i h_2$ or a descending chain $z_0 h_2 \ge z_1 h_2 \ge z_2 h_2 \ge \dots$, where in view of $z_{i+1} \notin z_i S$ $(i \in \omega)$ every inequality is strict, contradicting (2).

We claim that an element $z \in \mathcal{H}_1$ for which $h_1 < zh_2$ is unique. Assume that there exists $z' \in \mathcal{H}_1$ such that $h_1 < z'h_2$ and $z \ne z'$. Then $h_1 < z'z^{-1}(zh_2)$. Since ${}_SA \models \Phi_{L_3}(z'z^{-1})$, we have that there is $z_1 \in S$ such that $h_1 < z_1 h_2$, $z_1 \le z$, and $z_1 \le z'$. Hence, as we noted above, there exists $z_2 \in \mathcal{H}_1$ such that $h_1 \le z_2 h_2 \le z_1 h_2$. Hence we have $z_2 \le z$ and $z_2 \le z'$. As z and z' are the different elements, we have that either $z_2 < z$ or $z_2 < z'$, i.e., on the poset $\mathcal{H}_1$ the relation $\le$ is not coincide with equality, a contradiction.

Let $sh_1 < th_2$. We claim that there exists a unique $z \in \mathcal{H}_1$ such that $h_1 \le zh_2$ and $szh_2 \le th_2$. Since ${}_SA \models \Phi_{L_1}(s,t)$, we have that there is $z' \in S$ such that $sz'h_2 \le th_2$ and $h_1 \le z'h_2$. As proved above, there exists a unique $z \in \mathcal{H}_1$ such that $h_1 \le zh_2 \le z'h_2$. Then $szh_2 \le sz'h_2 \le th_2$.

Let $x \in X$,

$$X_x = \{y \in X \mid x \text{ is comparable with } y \text{ in the ordering } \le\},$$

and $s \in S$. We denote an element szh_y by s_y $(y \in X_x)$, where z is an element of $\mathcal{H}_1$ such that h_x is comparable with zh_y. As mentioned above, the element s_y is constructed uniquely. Then for all $x, y \in X$ and $s, t \in S$ condition (1) of Theorem 1.3 holds, i.e., ${}_SA$ is an S-poset free over the poset X.

Finally, suppose that ${}_SA$ is an S-poset free over the poset X. We claim that ${}_SA \models \Sigma_{\mathcal{F}r^{\ll}}$. It is clear that ${}_SA \models \Sigma_{\mathcal{F}r}$. By Theorem 1.3, ${}_SA = \coprod\limits_{x \in X} {}_SSx$, where ${}_SSx$ are the copies of an S-poset ${}_SS$. As in Theorem 1.3, we denote the copies of the elements $s \in S$ by s_x for all $x \in S$. Thus, condition (1) of Theorem 1.3 holds. Let $\rho = (s,t)$ and $i \in \{1,2\}$. As ${}_SS \models \Phi_r(\rho)$, we have that ${}_SA \models \Phi_r(\rho)$.

We claim that ${}_SA \models \Phi_{L_i}(\rho)$. Let $sk1_x \le sl1_y$ and for $i = 2$ either $sk \notin tS$ or $tl \notin sS$, where $x, y \in X$. By Theorem 1.3, $x \le y$ and $sk \le tl$. By assumption (2), there exists a maximal element r in the poset S such that $k \le r$, $sr \le tl$, and for $i = 2$ either $sr \notin tS$ or $tl \notin sS$. Then $\langle r, l\rangle \in L_i(\rho)$ and by assumption (4) $\langle r, l\rangle = \langle x^0, y^0\rangle w$ for some $w \in S$ and $\langle x^0, y^0\rangle \in L^i_\rho$. Hence $k1_x \le k1_y \le r1_y$, $\langle r1_y, l1_y\rangle = \langle x^0, y^0\rangle w1_y$, and for $i = 2$ either $sr \notin tS$ or $tl \notin sS$. Consequently, ${}_SA \models \Phi_{L_i}(\rho)$.

Let us claim that ${}_SA \models \Phi_{L_3}(s)$, where $s \in \mathcal{H}_1$. Let $k1_x \le l1_y$ and $k1_x \le sl1_y$ for some $k, l \in S$ and $x, y \in X$. By Theorem 1.3, $x \le y$, $k \le l$, and $k \le sl$. By assumption (2), there exists a maximal element r in the poset S such that $k \le r$, $r \le l$, and $r \le sl$. Then $\langle r, l\rangle \in L_3(s)$ and by assumption (5) $\langle r, l\rangle = \langle x^0, y^0\rangle w$ for some $w \in S$ and $\langle x^0, y^0\rangle \in L^3_s$. Hence $k1_x \le k1_y \le r1_y$, $\langle r1_y, l1_y\rangle = \langle x^0, y^0\rangle w1_y$. Consequently, ${}_SA \models \Phi_{L_3}(s)$.

We deduce that ${}_SA$ is a free S-poset over a poset X if and only if ${}_SA \models \Sigma_{\mathcal{F}r^{\ll}}$. Thus, the class $\mathcal{F}r^{\ll}$ is axiomatizable. $\square$

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