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Coupled thermal stresses analysis in the composite elastic-plastic cylinder

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Abstract. The present study is devoted to the set of boundary value problems in the frameworks of coupled thermoelastoplasticity under axial symmetry conditions for a composite circular cylinder. Throughout the paper the conventional Prandtl–Reuss elastic–plastic model generalised on the thermal effects is used. The yield stress is assumed by linear function of the temperature. The plastic potential is chosen in the form of Tresca yield criterion and the associated plastic flow rule is derived. The adding process of a heated cylinder to another is simulated. The coupled thermal stresses are calculated during processes of cooling and material unloading. The elastic-plastic borders positions are calculated and plastic flow domains are localized. Numerical results are graphically analysed.

Introduction

The modern mechanical engineering deals with the additive manufacturing technologies based on the adding new material at the high temperature gradient. The process of adding new parts of the material can be considered as a process of discrete material growth used in the technology of additive manufacturing. The mechanics of growing bodies [1–4] can be considered as a theoretical basis for solving such problems. In [5, 6], boundary value problems of the growth of heavy viscoelastic bodies were solved with the gravitational forces presence. The thermal state of a growing viscoelastic sphere was discussed in [7, 8]. The Assemblages made by the method of hot fitting have become the most widespread in technological practice for the stressed joints [9–13]. Usually such joints are carried out on the cylindrical surfaces of the assembly elements, when the outer part is heated before fitting and the inner one is cooled or remains at room temperature. Simplicity of the method of connection is coupled with the possibility to transmit significant loading pressure with different magnitudes and directions. The effective approximate engineering approaches and numerical simulation were developed to prescribe the stress strain state and material behaviour during the fitting process in the assembled nodes. The main drawback of the existing approaches for fitting process simulations is the insufficiently consistent consideration of the plastic flow and irreversible deformations in the assembled materials. The accumulation of irreversible deformations during plastic flow and subsequent unloading and cooling is significantly affect to the formation of the residual stresses and the final tightness of assembled part.

The solutions of the boundary value problem of stresses computations in circular compounds during the hot fitting process taking into account the plastic properties of the material was considered in [10–13]. The present study deals with a new solution obtained in the frameworks of plastic flow theory and under conditions of complete plasticity and repeated plastic flows during the cooling of the assembled material.

1. Boundary value problem statement. Thermoelastic equilibrium

Consider the boundary value problem in the frameworks of the thermal stresses theory during an assembly of two infinitely long hollow cylinders made from the same thermoelastoplastic material. At the referential time $t = 0$ the inner circular cylinder with inner and outer radii R_0 and R_1 respectively is heated under referential temperature T_1. The outer circular cylinder under temperature $T_2 > T_1$ with inner and outer radii R_1 and R_2 respectively. The referential displacements in the material of the cylinders are assumed to be zero. The inner cold cylinder is inserted into the heated outer cylinder. The stresses are accumulated providing tightness in the assembled parts as a result of the thermal conductivity process through the contact surface $r = R_1$. It is necessary to calculate the temperature field arising from heat conduction to determine the stress-strain states in the cylinders during the fitting. The heat conduction equation in the cylindrical coordinate system under conditions of axial symmetry can read by

$$r\dot{T} = \kappa(T_{,r} + rT_{,rr}). \tag{1}$$

wherein κ is the thermal diffusivity, overdot denotes the time derivative, index after the comma denotes partial derivative with respect to the spatial coordinate. We numerically integrate equation (1) under boundary and initial conditions

$$\begin{aligned} &T(r,0) = \begin{cases} T_1 & R_0 \le r \le R_1, \\ T_2 & R_1 < r \le R_2, \end{cases} \\ &T_{,r}(R_0,t) = 0, \\ &T_{,r}(R_2,t) = 0. \end{aligned} \tag{2}$$

Fig. 1 illustrates the temperature field distribution at different times in the assembled cylinders. Note that the conditions for an ideal thermal contact between the connected cylinders, i.e. the equality of temperatures and heat fluxes at the surface $r = R_1$ are automatically satisfied in the numerical solution of the heat equation (1) under boundary and initial conditions (2).

Deformations d_{ij} in the cylinders material are assumed infinitesimal and are separated into reversible part e_{ij} and irreversible p_{ij} one additively by formula

$$d_{ij} = e_{ij} + p_{ij}, \quad d_{rr} = u_{r,r}, \quad d_{\varphi\varphi} = r^{-1}u_r. \tag{3}$$

The stresses are fully determined by reversible deformations according to the Duhamel–Neumann law [14–16]

$$\begin{aligned} \sigma_{rr} &= (\lambda + 2\mu)e_{rr} + \lambda(e_{\varphi\varphi} + e_{zz}) - (3\lambda + 2\mu)\Delta, \\ \sigma_{\varphi\varphi} &= (\lambda + 2\mu)e_{\varphi\varphi} + \lambda(e_{rr} + e_{zz}) - (3\lambda + 2\mu)\Delta, \\ \sigma_{zz} &= (\lambda + 2\mu)e_{zz} + \lambda(e_{\varphi\varphi} + e_{rr}) - (3\lambda + 2\mu)\Delta. \end{aligned} \tag{4}$$

Here λ, μ are the Lame parameters. The thermal expansion function Δ for considered problem can be furnished by

$$\Delta(r,0) = \alpha(T(r,t) - T(r,0)) = \begin{cases} \alpha(T(r,t) - T_1) & R_0 \le r \le R_1, \\ \alpha(T(r,t) - T_2) & R_1 < r \le R_2. \end{cases} \tag{5}$$

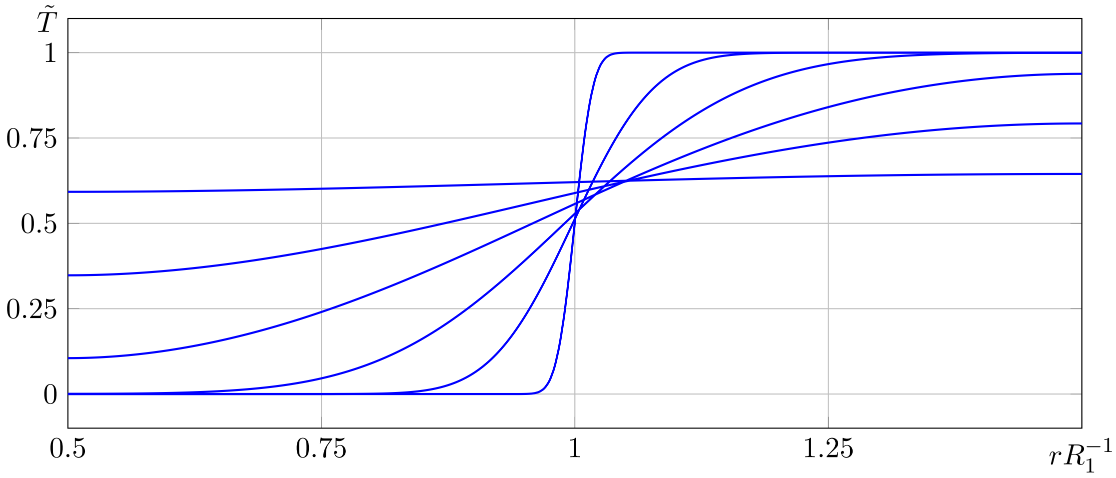

Figure 1. Temperature field in the assembled cylinders. $\tilde{T} = (T - T_1)(T_2 - T_1)^{-1}$.

Equation (5) gives the zero deformations and discplacements at time of the cylinder connection (fitting time) $t = 0$.

The equilibrium equation in the cylindrical coordinate system under conditions of axial symmetry reads as

$$\sigma_{rr,r} + r^{-1}(\sigma_{rr} - \sigma_{\varphi\varphi}) = 0. \tag{6}$$

The solution of the coupling equations (1), (4), (3), (6) under conditions (2), (5) and $p_{ij} = 0$ describe the thermoelastic equilibrium of assembled cylinders.

$$\begin{aligned} \sigma_{rr}^{(i)} &= -\frac{2\omega}{r^2}\int_{R_0}^{r} \Delta(\rho,t)\rho d\rho + A_i(t) + \frac{B_i(t)}{r^2}, \\ \sigma_{\varphi\varphi}^{(i)} &= \frac{2\omega}{r^2}\int_{R_0}^{r} \Delta(\rho,t)\rho d\rho + A_i(t) - \frac{B_i(t)}{r^2} - 2\omega\Delta(r,t), \\ \sigma_{zz}^{(i)} &= \frac{\lambda A_i(t)}{(\lambda+\mu)} - 2\omega\Delta(r,t). \end{aligned} \tag{7}$$

Index (i) denote the inner (1) or outer (2) cylinders. Unknown time dependent functions $A_i(t)$, $B_i(t)$ can be derived from the system of boundary conditions

$$\begin{aligned} &\sigma_{rr}^{(1)}(R_0,t) = 0, \quad \sigma_{rr}^{(1)}(R_1,t) = \sigma_{rr}^{(2)}(R_1,t), \\ &\sigma_{rr}^{(2)}(R_2,t) = 0, \quad u_r^{(1)}(R_1,t) = u_r^{(2)}(R_1,t). \end{aligned} \tag{8}$$

Finally the solution of equations (8) can be obtained in the following form

$$A_i = \frac{2\omega}{(R_2^2 - R_0^2)}\int_{R_0}^{R_2} \Delta(\rho,t)\rho d\rho, \quad B_i = -\frac{2\omega R_0^2}{(R_2^2 - R_0^2)}\int_{R_0}^{R_2} \Delta(\rho,t)\rho d\rho. \tag{9}$$

2. Plastic flow in the outer cylinder

The plastic flow occurs on the contact surface of the outer cylinder and is propagated with the temperature field aligning inside the assembled material. We note that in some cases under a sufficiently high initial temperature decreasing a plastic flow can occur on the contact surface of the inner cylinder, but numerous simulations have shown that the level of plastic deformation in these cases is small in compared with deformations arising from the contact pressure and does not cause a noticeable influence on the distribution of residual stresses and deformations.

Let the time $t = t_p^{(2)}$ is the time of the plastic flow beginning on the inner surface of the outer cylinder. The plastic potential in the form of Tresca yield criterion is more preferable for considered problem

$$\sigma_{\varphi\varphi}^{(2)} - \sigma_{rr}^{(2)} = 2k(r,t). \tag{10}$$

At a time $t > t_p^{(2)}$ the plastic flow domain $R_1 < r < a_2(t)$ in the outer cylinder, where $a_2(t)$ denotes the elastic plastic border. Equation for stresses computation can be obtained by integration of the equilibrium equation (6) under condition (10):

$$\begin{aligned}
\sigma_{rr}^{(2)} &= 2\int_{R_0}^{r} \frac{k(\rho,t)}{\rho} d\rho + C_2(t), \\
\sigma_{\varphi\varphi}^{(2)} &= 2\int_{R_0}^{r} \frac{k(\rho,t)}{\rho} d\rho + C_2(t) + 2k(r,t), \\
\sigma_{zz}^{(2)} &= \frac{\lambda}{(\lambda+\mu)}\left(C_2(t) + 2\int_{R_0}^{r} \frac{k(\rho,t)}{\rho} d\rho + k(r,t)\right) - \gamma\Delta(r,t).
\end{aligned} \tag{11}$$

Note that the lower limit of integration in the equations (7), (11) which valid for the outer cylinder is the inner radius ($R_0 < R_2$) of the inner cylinder. For this case the functions $A_i(t),\ B_i(t),\ C_i(t)...$ have a simplest form [17–20].

Equations for radial displacement and plastic deformation in the plastic flow domain $R_1 < r < a_2(t)$ read by

$$\begin{aligned}
u_r^{(2)} &= \frac{C_2(t)r}{2(\lambda+\mu)} + \frac{D_2(t)}{r} + \frac{r}{(\lambda+\mu)}\int_{R_0}^{r} \frac{k(\rho,t)}{\rho} d\rho + \frac{\gamma}{\mu r}\int_{R_0}^{r} \Delta(\rho,t)\rho d\rho, \\
p_{rr}^{(2)} &= \frac{(\lambda+2\mu)k(r,t)}{2\mu(\lambda+\mu)} + \frac{\gamma\Delta(r,t)}{2\mu} - \frac{\gamma}{\mu r^2}\int_{R_0}^{r} \Delta(\rho,t)\rho d\rho - \frac{D_2(t)}{r^2}, \\
p_{zz}^{(2)} &= 0, \quad p_{\varphi\varphi}^{(2)} = -p_{rr}^{(2)}.
\end{aligned} \tag{12}$$

Unknown time dependent functions in virtue of the condition (8) and continuity conditions for radial stresses and displacement on the elastic plastic border a_2. Hereafter solutions for these functions are not shown because of their cumbersomeness. The position of the elastic plastic border is calculated by a numerical solution of equation $p_{rr}^{(2)}(a_2,t) = 0$, for a certain time $t > t_p^{(2)}$.

3. Plastic flow in the inner cylinder

The further temperature field rearranging in the assembled cylinders can lead to plastic flow on the inner surface of the inner cylinder at time $t = t_p^{(1)}$ [21–24]. Then the stresses keep the following form of the Tresca yield criterion [17, 18, 21]:

$$\sigma_{rr}^{(1)} - \sigma_{zz}^{(1)} = 2k(r,t). \tag{13}$$

Consequently for a time $t > t_p^{(1)}$ it is possible a plastic flow domain development $R_0 < a_1(t)$ in the material of the inner cylinder. The elastic domain is separated from the domain of irreversible deformation by the elastic plastic border $a_1(t)$. We can derived the plastic incompressibility conditions from the plastic flow rule associated with the yield criterion (13)

$$p_{zz} + p_{rr} = 0, \quad p_{\varphi\varphi} = 0; \tag{14}$$

The relations between the radial components of the reversible and irreversible strain tensors and the radial displacement are obtained from equations (3), (4), (14), (13) in following form

$$e_{rr} = \frac{1}{2}\left(u_{r,r} + \frac{k}{\mu}\right), \quad p_{rr} = \frac{1}{2}\left(u_{r,r} - \frac{k}{\mu}\right), \tag{15}$$

Then the differential equations for the radial displacement furnishes taking into account equilibrium equation (6) and equation (15) as follows

$$(ru_{r,r})_{,r} - \frac{\eta^2 u_r}{r} + \frac{(rk)_{,r}}{(\lambda+\mu)} - \frac{\gamma r\Delta_{,r}}{\mu} = 0, \quad \eta = \sqrt{\frac{(\lambda+2\mu)}{(\lambda+\mu)}}. \tag{16}$$

The function of the radial displacement is the solution of the equation (16) for the plastic flow domain

$$\begin{gathered} u_r = \frac{\gamma}{2\mu\eta}\left(\frac{(\eta+1)}{r^\eta}\int_{R_1}^r \rho^\eta \Delta(\rho,t)d\rho + (\eta-1)r^\eta \int_{R_1}^r \frac{\Delta(\rho,t)}{\rho^\eta}d\rho\right) - \\ -\frac{1}{2(\lambda+\mu)}\left(r^\eta \int_{R_1}^r \frac{k(\rho,t)}{\rho^\eta}d\rho + \frac{1}{r^\eta}\int_{R_1}^r \rho^\eta k(\rho,t)d\rho\right) + C(t)r^\eta + \frac{D(t)}{r^\eta}. \end{gathered} \tag{17}$$

The plastic deformation then is found by virtue of equations (15) as

$$\begin{gathered} p_{rr} = \frac{\eta}{4(\lambda+\mu)}\left(\frac{1}{r^{(\eta+1)}}\int_{R_1}^r \rho^\eta k(\rho,t)d\rho - r^{(\eta-1)}\int_{R_1}^r \frac{k(\rho,t)}{\rho^\eta}d\rho\right) + \frac{\gamma\Delta(r,t)}{2\mu} + \\ +\frac{\gamma}{4\mu}\left(r^{(\eta-1)}\int_{R_1}^r \frac{\Delta(\rho,t)}{\rho^\eta}d\rho - \frac{(\eta+1)}{r^{(\eta+1)}}\int_{R_1}^r \rho^\eta \Delta(\rho,t)d\rho\right) - \frac{\eta^2 k(r,t)}{2\mu} + \\ +\frac{\eta r^{(\eta-1)}C(t)}{2} - \frac{\eta D(t)}{2r^{(\eta+1)}}, \\ p_{zz} = -p_{rr}, \quad p_{\varphi\varphi} = 0. \end{gathered} \tag{18}$$

Let derive the equations for stresses (4) in the plastic flow domain by using equations (15), (17) and (18)

$$\begin{gathered} \sigma_{rr} = \sigma_{zz} + 2k(r,t) = \nu_1 r^{(\eta-1)}C(t) - \frac{\nu_2 D(t)}{r^{(\eta+1)}} - \\ -\frac{1}{2(\lambda+\mu)}\left(\nu_1 r^{(\eta-1)}\int_{R_1}^r \frac{k(\rho,t)}{\rho^\eta}d\rho - \frac{\nu_2}{r^{(\eta+1)}}\int_{R_1}^r \rho^\eta k(\rho,t)d\rho\right) + \\ +\frac{\gamma}{2\eta\mu}\left((\eta-1)\nu_1 r^{(\eta-1)}\int_{R_1}^r \frac{\Delta(\rho,t)}{\rho^\eta}d\rho - \frac{(\eta+1)\nu_2}{r^{(\eta+1)}}\int_{R_1}^r \rho^\eta \Delta(\rho,t)d\rho\right), \\ \nu_1 = (\eta\lambda + \lambda + \eta\mu), \quad \nu_1 = (\eta\lambda - \lambda + \eta\mu), \\ \sigma_{\varphi\varphi} = \upsilon_1 r^{(\eta-1)}C(t) + \frac{\upsilon_2 D(t)}{r^{(\eta+1)}} - \gamma\Delta(r,t) - \frac{\lambda k(r,t)}{(\lambda+\mu)} - \\ -\frac{1}{2(\lambda+\mu)}\left(\upsilon_1 r^{(\eta-1)}\int_{R_1}^r \frac{k(\rho,t)}{\rho^\eta}d\rho + \frac{\upsilon_2}{r^{(\eta+1)}}\int_{R_1}^r \rho^\eta k(\rho,t)d\rho\right) + \\ +\frac{\gamma}{2\eta\mu}\left((\eta-1)\upsilon_1 r^{(\eta-1)}\int_{R_1}^r \frac{\Delta(\rho,t)}{\rho^\eta}d\rho + \frac{(\eta+1)\upsilon_2}{r^{(\eta+1)}}\int_{R_1}^r \rho^\eta \Delta(\rho,t)d\rho\right), \\ \upsilon_1 = (\lambda + \eta\lambda + 2\mu), \quad \upsilon_2 = (\lambda - \eta\lambda + 2\mu). \end{gathered} \tag{19}$$

4. Complete plasticity in the inner cylinder

Note also that there is the possibility of the complete plasticity state in the frameworks of Tresca yield criterion [21, 22, 25, 26]. For this case we have two valid plastic flow conditions

$$\begin{cases} \sigma_{rr}^{(1)} - \sigma_{zz}^{(1)} = 2k(r,t), \\ \sigma_{rr}^{(1)} - \sigma_{\varphi\varphi}^{(1)} = 2k(r,t). \end{cases} \tag{20}$$

The stresses in the complete plasticity domain ($R_0 < r < b_1(t)$) are furnished by following relations

$$\begin{aligned} \sigma_{rr}^{(1)} &= -2\int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho + E(t), \\ \sigma_{\varphi\varphi}^{(1)} &= \sigma_{zz}^{(1)} - 2\int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho - 2k(r,t) + E(t). \end{aligned} \tag{21}$$

The plastic incompressibility condition $p_{rr} + p_{zz} + p_{\varphi\varphi} = 0$ implies the coupling between the total and reversible deformations as follows

$$u_{r,t} + \frac{u_r^{(1)}}{r} = e_{rr} + e_{\varphi\varphi} + e_{zz}. \tag{22}$$

Substitute the elastic deformation according to (4) and (21) in equation (22) and integrate obtained equation in the complete plasticity domain

$$\begin{aligned} u_r^{(1)} = &-\frac{1}{(3\lambda + 2\mu)}\left(\frac{1}{r}\int_{R_1}^{r} k(\rho,t)\rho d\rho + 3r\int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho - \frac{3rE(t)}{2}\right) + \\ &+ \frac{F(t)}{r} + \frac{3}{r}\int_{R_1}^{r} \Delta(\rho,t)\rho d\rho. \end{aligned} \tag{23}$$

Plastic deformations functions taking into account equation (23) in the domain ($R_1 < r < b_1(t)$) reads by

$$\begin{aligned} p_{rr}^{(1)} &= \frac{1}{(3\lambda + 2\mu)}\left(\frac{1}{r^2}\int_{R_1}^{r} k(\rho,t)\rho d\rho - \int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho + \frac{E(t)}{2}\right) - \\ &\quad - \frac{2k(r,t)}{\gamma} - \frac{3}{r^2}\int_{R_1}^{r} \Delta(\rho,t)\rho d\rho + 2\Delta(r,t) - \frac{F(t)}{r^2}, \\ p_{\varphi\varphi}^{(1)} &= -\frac{1}{(3\lambda + 2\mu)}\left(\frac{1}{r^2}\int_{R_1}^{r} k(\rho,t)\rho d\rho + \int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho - \frac{E(t)}{2}\right) + \\ &\quad + \frac{k(r,t)}{\gamma} + \frac{3}{r^2}\int_{R_1}^{r} \Delta(\rho,t)\rho d\rho - \Delta(r,t) + \frac{F(t)}{r^2}, \\ p_{zz}^{(1)} &= \frac{1}{(3\lambda + 2\mu)}\left(2\int_{R_1}^{r} \frac{k(\rho,t)}{\rho} d\rho - E(t)\right) + \frac{k(r,t)}{\gamma} - \Delta(r,t). \end{aligned} \tag{24}$$

The satisfaction of the present Tresca yield criterion forms (10), (13) and (20) depends on the cylinder size and referential temperature gradient. Fig. 2 illustrates the thermal stresses in the assembled cylinders after full temperature equalization with the complete plasticity domain in the inner cylinder.

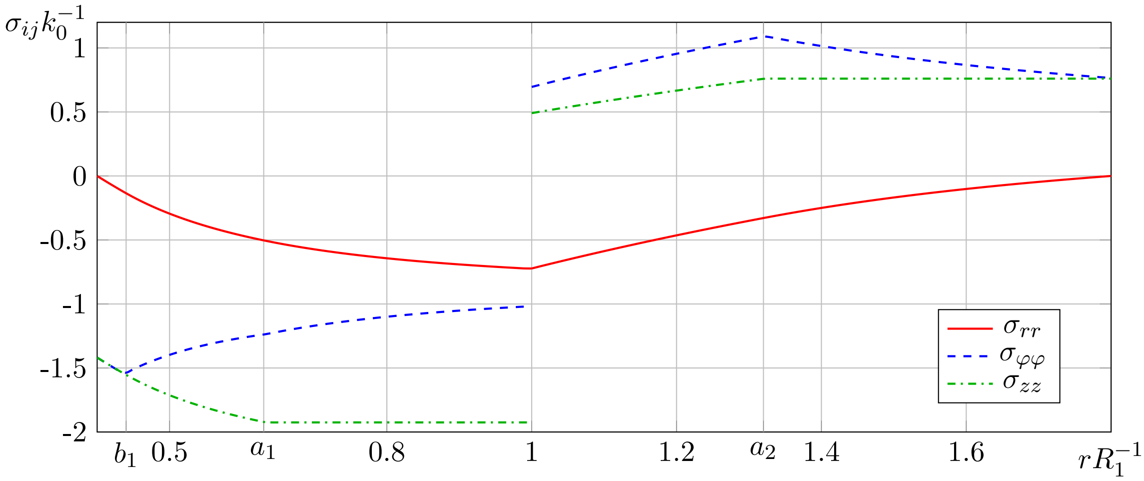

Figure 2. Thermal stresses in the composite cylinder under plastic flow. $R_0R_1^{-1} = 0.4$, $R_2R_1^{-1} = 1.8$.

5. Cooling to the referential temperature

Let consider the cooling process of the cylinders. We assume that the heat conduction with the external continuum is slow and the temperature field in the material of the cylinders corresponds to a uniform distribution slowly decreasing to the referential temperature of the inner cylinder. Thus the compressive stress $\sigma_{zz}^{(1)}$ during the cooling should decrease. Along with this, the value of the yield stress in the material of both cylinders increases. Consequently, the unloading occurs in the material of the inner cylinder during cooling (similar cases were discussed in [27–30]). The stress strain state of the outer cylinder due to increasing yield stress is also satisfies the unloading conditions for a certain temperature diapason. However a repeated plastic flow causing can begin as a cause of a high stress level $\sigma_z^{(2)}z$ corresponding to a stretching of the outer cylinder along the z axis.

Introduce notations $p_{ij}^{(1)}$ and $p_{ij}^{(2)}$ for the irreversible deformations of the cylinders computing at the time of final temperature equalization. These functions are vanished in elastic domains [31, 32]. Differential equation for the radial components of the stress tensor can be obtained from equations (4), (3), (6) in the form

$$\frac{\tau}{r}\left(p_{rr}^{(i)} - p_{\varphi\varphi}^{(i)} - rp_{\varphi\varphi,r}^{(i)}\right) - \frac{2\mu\lambda p_{zz,r}^{(i)}}{(\lambda+2\mu)} - 2\omega\Delta_{,r} - 3\sigma_{rr,r}^{(i)} - r\sigma_{rr,rr}^{(i)} = 0, \tag{25}$$

where $\tau = \dfrac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}$.

Integration of the equation (25) under condition $p_{zz}^{(1)} = -(p_{rr}^{(1)} + p_{\varphi\varphi}^{(1)})$, allows us to express the stresses in the inner cylinder by formulae

$$\sigma_{rr}^{(1)} = \frac{\tau}{2}\Big(\int_{R_0}^{r}\frac{p_{rr}^{(1)}(\rho)}{\rho}d\rho - \int_{R_0}^{r}\frac{p_{\varphi\varphi}^{(1)}(\rho)}{\rho}d\rho\Big) - \frac{2\omega}{r^2}\int_{R_0}^{r}\Delta(\rho,t)\rho d\rho - \\ -\frac{2\mu^2}{(\lambda+2\mu)r^2}\Big(\int_{R_0}^{r}p_{rr}^{(1)}(\rho)\rho d\rho + \int_{R_0}^{r}p_{\varphi\varphi}^{(1)}(\rho)\rho d\rho\Big) + X_1(t) + \frac{Y_1(t)}{r^2},$$

$$\sigma_{\varphi\varphi}^{(1)} = \frac{\tau}{2}\Big(\int_{R_0}^{r}\frac{p_{rr}^{(1)}(\rho)}{\rho}d\rho - \int_{R_0}^{r}\frac{p_{\varphi\varphi}^{(1)}(\rho)}{\rho}d\rho\Big) + \frac{2\omega}{r^2}\int_{R_0}^{r}\Delta(\rho,t)\rho d\rho + \\ +\frac{2\mu^2}{(\lambda+2\mu)r^2}\Big(\int_{R_0}^{r}p_{rr}^{(1)}(\rho)\rho d\rho + \int_{R_0}^{r}p_{\varphi\varphi}^{(1)}(\rho)\rho d\rho\Big) + X_1(t) - \frac{Y_1(t)}{r^2} + \\ +\frac{2\lambda\mu p_{rr}^{(1)}(r)}{(\lambda+2\mu)} - 2\omega\Delta(r,t) - 2\mu p_{\varphi\varphi}^{(1)}(r), \tag{26}$$

$$\sigma_{zz}^{(1)} = \frac{2\lambda\mu}{(\lambda+2\mu)}\Big(\int_{R_0}^{r}\frac{p_{rr}^{(1)}(\rho)}{\rho}d\rho - \int_{R_0}^{r}\frac{p_{\varphi\varphi}^{(1)}(\rho)}{\rho}d\rho\Big) + \tau p_{rr}^{(1)}(r) + \\ +2\mu p_{\varphi\varphi}^{(1)} + \frac{\lambda X_1(t)}{(\lambda+\mu)} - 2\omega\Delta(r,t).$$

The radial displacement in this case is calculated as

$$u_r^{(1)} = \frac{\omega}{\mu r}\int_{R_0}^{r}\Delta(\rho,t)\rho d\rho + \frac{rX_1(t)}{2(\lambda+\mu)} - \frac{Y_1(t)}{2\mu r} + \frac{\mu r}{(\lambda+2\mu)}\Big(\int_{R_0}^{r}\frac{p_{rr}^{(1)}(\rho)}{\rho}d\rho - \\ -\int_{R_0}^{r}\frac{p_{\varphi\varphi}^{(1)}(\rho)}{\rho}d\rho + \frac{1}{r^2}\int_{R_0}^{r}p_{rr}^{(1)}(\rho)\rho d\rho + \frac{1}{r^2}\int_{R_0}^{r}p_{\varphi\varphi}^{(1)}(\rho)\rho d\rho\Big). \tag{27}$$

The stresses in the outer cylinder in virtue of conditions $p_{rr}^{(2)} = -p_{\varphi\varphi}^{(2)}$ and $p_{zz}^{(2)} = 0$ reads by

$$\sigma_{rr}^{(2)} = \tau\int_{R_1}^{r}\frac{p_{rr}^{(2)}(\rho)}{\rho}d\rho - \frac{2\omega}{r^2}\int_{R_0}^{r}\Delta(\rho,t)\rho d\rho + X_2(t) + \frac{Y_2(t)}{r^2},$$

$$\sigma_{\varphi\varphi}^{(2)} = \tau\Big(\int_{R_1}^{r}\frac{p_{rr}^{(2)}(\rho)}{\rho}d\rho + p_{rr}^{(2)}(r)\Big) + X_2(t) - \\ -\frac{2\omega}{r^2}\Big(\int_{R_0}^{r}\Delta(\rho,t)\rho d\rho - r^2\Delta(r,t)\Big) - \frac{Y_2(t)}{r^2}, \tag{28}$$

$$\sigma_{zz}^{(2)} = \frac{2\lambda\mu}{(\lambda+2\mu)}\Big(2\int_{R_1}^{r}\frac{p_{rr}^{(2)}(\rho)}{\rho}d\rho + p_{rr}^{(2)}(r)\Big) + \frac{\lambda X_2(t)}{(\lambda+\mu)} - 2\omega\Delta(r,t).$$

The displacement in the outer cylinder in the time is satisfied to following equation

$$u_{rr}^{(2)} = \frac{2\mu r}{(\lambda+2\mu)}\int_{R_1}^{r}\frac{p_{rr}^{(2)}(\rho)}{\rho}d\rho + \frac{\omega}{r}\int_{R_0}^{r}\Delta(\rho,t)\rho d\rho + \frac{rX_2(t)}{2(\lambda+\mu)} - \frac{Y_2(t)}{2\mu r}. \tag{29}$$

Unknown time dependent functions $X_i(t)$, $Y_i(t)$ can be found by boundary conditions (8).

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6. Repeated plastic flow

The stress strain state at a time $t = t_r$ can rich the yield surface on the inner surface of the outer cylinder as the assembled construction is cooling down. In this case Tresca yield criterion is valid in the following form

$$\sigma_{zz}^{(2)} - \sigma_{rr}^{(2)} = 2k. \tag{30}$$

From this time $t > t_r$ the repeated plastic flow domain is propagated in the outer cylinder $(R_1 < r < b_2(t))$. This repeated irreversible deformations are caused by cylinder stretching under cooling [23, 28].

To compute the stress-strain state parameters in this domain it is necessary to take into account the accumulated repeated irreversible deformations $p_{rr}^{(2)}$. Thus the equilibrium equation in terms of displacement is furnished by

$$(ru_{r,r})_{,r} - \frac{\eta^2 u_r}{r} - \frac{(rk + rp_{rr}^{(2)})_{,r}}{(\lambda+\mu)} - \frac{\gamma r \Delta_{,r}}{\mu} = 0, \tag{31}$$

We can solve the equation (31) and rewrite the displacement function by

$$\begin{aligned}\tilde{u}_r = &\frac{\gamma}{2\mu\eta}\left(\frac{(\eta+1)}{r^\eta}\int_{R_1}^r \rho^\eta \Delta(\rho,t)d\rho + (\eta-1)r^\eta \int_{R_1}^r \frac{\Delta(\rho,t)}{\rho^\eta}d\rho\right) - \\ &+\frac{1}{2(\lambda+\mu)}\left(r^\eta \int_{R_1}^r \frac{k(\rho,t)}{\rho^\eta}d\rho + \frac{1}{r^\eta}\int_{R_1}^r \rho^\eta k(\rho,t)d\rho\right) + \\ &+\frac{\mu}{2(\lambda+\mu)}\left(r^\eta \frac{(\eta+2)}{\eta}\int_{R_1}^r \frac{p_{rr}^{(2)}(\rho)}{\rho^\eta}d\rho + \frac{1}{r^\eta}\frac{(\eta-2)}{\eta}\int_{R_1}^r \rho^\eta p_{rr}^{(2)}(\rho)d\rho\right) + \\ &+P_2(t)r^\eta + \frac{Q_2(t)}{r^\eta}.\end{aligned} \tag{32}$$

Hereafter the tilde denote the functions determining in the repeated plastic flow domain. The plastic deformations by virtue of the solution (32) are calculated by formula

$$\begin{aligned}\tilde{p}_{rr} = &\frac{\eta}{4(\lambda+\mu)}\left(r^{(\eta-1)}\int_{R_1}^r \frac{k(\rho,t)}{\rho^\eta}d\rho - \frac{1}{r^{(\eta+1)}}\int_{R_1}^r \rho^\eta k(\rho,t)d\rho\right) + \frac{\gamma\Delta(r,t)}{2\mu} + \\ &+\frac{\gamma}{4\mu}\left(r^{(\eta-1)}\int_{R_1}^r \frac{\Delta(\rho,t)}{\rho^\eta}d\rho - \frac{(\eta+1)}{r^{(\eta+1)}}\int_{R_1}^r \rho^\eta \Delta(\rho,t)d\rho\right) + \frac{\eta^2 k(r,t)}{2\mu} + \\ &+\frac{\mu\eta}{4(\lambda+\mu)}\left(r^{(\eta-1)}\frac{(\eta+2)}{\eta}\int_{R_1}^r \frac{p_{rr}^{(2)}(\rho)}{\rho^\eta}d\rho - \frac{1}{r^{(\eta+1)}}\frac{(\eta+2)}{\eta}\int_{R_1}^r \rho^\eta p_{rr}^{(2)}(\rho)d\rho\right) + \\ &+\frac{\eta r^{(\eta-1)}P_2(t)}{2} - \frac{\eta Q_2(t)}{2r^{(\eta+1)}} + \frac{\eta^2 p_{rr}^{(2)}(r)}{2}, \\ \tilde{p}_{zz} = &-\tilde{p}_{rr}(r,t) + p_{rr}^{(2)}(r), \quad \tilde{p}_{\varphi\varphi} = -p_{rr}^{(2)}(r).\end{aligned} \tag{33}$$

The components of the stress tensor are derived by the known displacement (32) and plastic deformations (33) as

$$\begin{aligned}
&\tilde{\sigma}_{rr} = -\tilde{\sigma}_{zz} - 2k(r,t) = \nu_1 r^{(\eta-1)} P_2(t) - \frac{\nu_2 Q_2(t)}{r^{(\eta+1)}} - \\
&+\frac{1}{2(\lambda+\mu)}\left(\nu_1 r^{(\eta-1)} \int_{R_1}^{r} \frac{k(\rho,t)}{\rho^\eta} d\rho - \frac{\nu_2}{r^{(\eta+1)}} \int_{R_1}^{r} \rho^\eta k(\rho,t) d\rho\right) + \\
&+\frac{\gamma}{2\eta\mu}\left((\eta-1)\nu_1 r^{(\eta-1)} \int_{R_1}^{r} \frac{\Delta(\rho,t)}{\rho^\eta} d\rho - \frac{(\eta+1)\nu_2}{r^{(\eta+1)}} \int_{R_1}^{r} \rho^\eta \Delta(\rho,t) d\rho\right) + \\
&+\frac{\mu}{2(\lambda+\mu)}\left(\nu_1 r^{(\eta-1)} \frac{(\eta+2)}{\eta} \int_{R_1}^{r} \frac{p_{rr}^{(2)}(\rho)}{\rho^\eta} d\rho - \frac{\nu_2}{r^{(\eta+1)}} \frac{(\eta-2)}{\eta} \int_{R_1}^{r} \rho^\eta p_{rr}^{(2)}(\rho) d\rho\right), \\
&\nu_1 = (\eta\lambda + \lambda + \eta\mu), \quad \nu_1 = (\eta\lambda - \lambda + \eta\mu), \\
&\tilde{\sigma}_{\varphi\varphi} = \upsilon_1 r^{(\eta-1)} P_2(t) + \frac{\upsilon_2 Q_2(t)}{r^{(\eta+1)}} - \gamma\Delta(r,t) + \frac{\lambda k(r,t)}{(\lambda+\mu)} + \gamma p_{rr}^{(2)}(r) + \\
&+\frac{1}{2(\lambda+\mu)}\left(\upsilon_1 r^{(\eta-1)} \int_{R_1}^{r} \frac{k(\rho,t)}{\rho^\eta} d\rho + \frac{\upsilon_2}{r^{(\eta+1)}} \int_{R_1}^{r} \rho^\eta k(\rho,t) d\rho\right) + \\
&\frac{\mu}{2(\lambda+\mu)}\left(\nu_1 r^{(\eta-1)} (\eta+2) \int_{R_1}^{r} \frac{p_{rr}^{(2)}(\rho)}{\rho^\eta} d\rho + \frac{\nu_2}{r^{(\eta+1)}} (\eta-2) \int_{R_1}^{r} \rho^\eta p_{rr}^{(2)}(\rho) d\rho\right) + \\
&+\frac{\gamma}{2\eta\mu}\left((\eta-1)\upsilon_1 r^{(\eta-1)} \int_{R_1}^{r} \frac{\Delta(\rho,t)}{\rho^\eta} d\rho + \frac{(\eta+1)\upsilon_2}{r^{(\eta+1)}} \int_{R_1}^{r} \rho^\eta \Delta(\rho,t) d\rho\right), \\
&\upsilon_1 = (\lambda + \eta\lambda + 2\mu), \quad \upsilon_2 = (\lambda - \eta\lambda + 2\mu).
\end{aligned} \tag{34}$$

Unknown time dependent function $X_i(t)$, $Y_i(t)$, $P_2(t)$, $Q_2(t)$ can be found from the linear system of the boundary conditions (8) and continuity conditions of the radial stress and displacement at the boundary of the repeated plastic flow domain $b_2(t)$. The location of this border is numerically calculated as the solution of the equation $\tilde{p}_{rr}(b_2,t) = p_{rr}^{(2)}(r)$. The repeated plastic flow is propagated until the full cooling assembled cylinders to the referential temperature. After that the stress strain state riches the neutral loading state. On the Figure 3 the residual stresses are shown at the time of full cooling down. The neutral loading state as we can see on the Fig. 3 occupies the domain $R_1 < r < b_2$ with valid yield criterion $\tilde{\sigma}_{zz} - \tilde{\sigma}_{rr} = 2k_0$ and constant irreversible deformations.

Concluding remarks

The sequence of the considered boundary value problems may be violated by the cylinder size, material properties and referential temperature gradient. The extremal influence on the final contact pressure is induced by the irreversible deformation rate under yield criterion in form $\sigma_{zz} - \sigma_{rr} = 2k$. The dependence of the yield stress on temperature makes it possible to predict lower values of the contact pressure contrary to studies without the plastic properties accounting or with the constant yield stress. Therefore, to increase the value of the contact pressure it is necessary to reduce the initial temperatures in both parts of the assembled construction. Obviously, in this case the yield stress have the greatest value and, consequently, the level of the contact pressure is higher.

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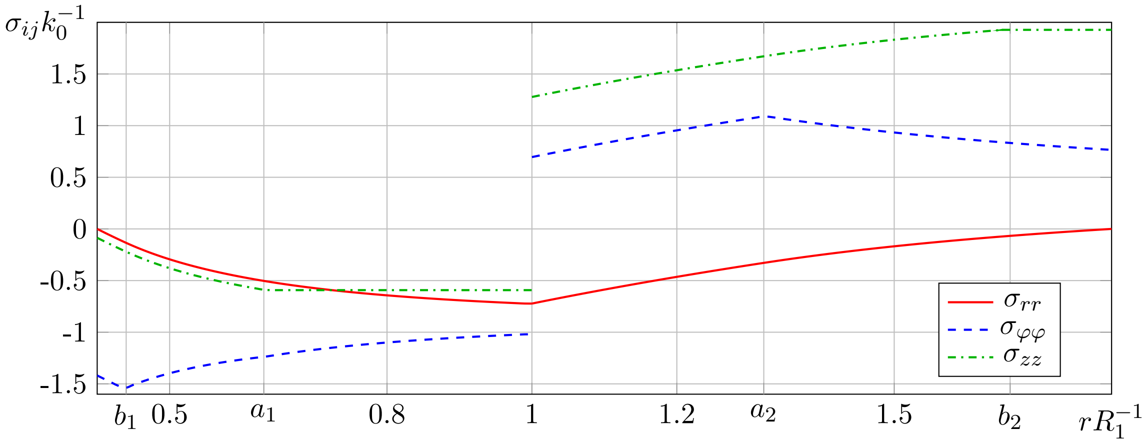

Figure 3. Thermal stresses in composite cylinder under cooling. $R_0R_1^{-1} = 0.4$, $R_2R_1^{-1} = 1.8$.

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