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On the Modeling of the Shrink Fit Technology

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Abstract—A solution is given of a one-dimensional problem of the theory of thermal stresses which simulates the hot shrink fit of a cylindrical clutch on a cylindrical shaft. The distinguished feature in the statement of the problem is taking into account the originating and developing plastic flow of the material of the assembly components due to the nonstationarity of the temperature field and the dependence of the yield material strength on the temperature. It is shown that irreversible deformation may significantly reduce the level of the final residual stresses providing the desired tightness.

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INTRODUCTION

Among the pressure couplings, the assembling realized by means of hot shrink fit has become the most widespread in the technological practice [1]. Most often this assembling is carried out along the cylindrical surfaces of the assembly components, when the external member is heated before the fit, whereas the internal member either cools down or remains at the room temperature. In addition, the simplicity of the assembly method is supplemented by the possibility of transferring the substantial in absolute value and different in direction loading forces.

To calculate the fit process and its result, some efficient approximate engineering approaches turned into methods have been developed [2]; also numerical calculations have been carried out to study the levels and distributions of stresses in the structural components assembled in such manner [3]. In our opinion, the main shortcoming of the existing approaches to simulating of both the fit process and its result is connected with the insufficiently consistent account for the parameters of the plastic flow in the assembly materials. The development of the plastic flow with accumulation of the irreversible deformations and its subsequent slowing down under unloading and cooling down significantly influences the formation of the field of residual stresses which determines the final tightness.

Let us consider a simple one-dimensional problem concerning fitting of a cylindrical clutch on a cylindrical shaft. In its setting we will not strictly follow the technical conditions of the technology, trespassing the limits of the recommended heating temperatures, and will not take into account the contact slipping friction and other such effects accompanying the process. In this way, on one hand, we idealize the technological process to the extent that it can be represented in the form of a model problem; on the other hand, we will not restrict the conditions of the model problem by the recommended technological conditions of assembling.

Assume that in the neighborhood of the contact surface the material of the assembly components may deform intensely and, most importantly, irreversibly. Let us connect the origination and development of

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the plastic flow with the increase of contact stresses in the conditions of a nonstationary heat exchange process under the decreased yield strength of the assembly materials due to initial heating.

The theory of thermal plasticity [4] including the thermal stress theory in the flow conditions allowed us to obtain some useful answers to a few questions of technological practice [5–7]. There are clear perspectives for its development in this direction. This publication is intended to substantiate this.

1. STATEMENT OF THE PROBLEM

Let a cylindrical clutch (a long hollow cylinder) having the dimensions $R \le r \le R_1$ and heated up to the temperature T_* be fit on a cylindrical shaft of radius R at the room temperature T_0.

If the edge effects are not taken into account then the subsequent problem of heat transfer and deformation can be considered as one-dimensional. In this case, for the temperature field created by the contact of the heated clutch and the shaft, we have in the cylindrical coordinate system r, φ, and z the heat-transfer equation in the form

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right). \tag{1.1}$$

Here $T(r,t)$ is the current temperature and a is the temperature conductivity coefficient. In order to simplify the problem, we assume the material of the clutch and the shaft to be the same.

The equation (1.1) should be supplemented by the boundary and initial conditions. As the latter, we take $T(r,0) = T_0$ for $r \le R$ and $T(r,0) = T_*$ for $r \ge R$. At $r = R_1$ we have the contact of the heated body with the environment (air); therefore,

$$\left.\frac{\partial T}{\partial t}\right|_{r=R_1} = \chi T_0, \tag{1.2}$$

where the constant χ is the coefficient of the heat transfer from the hot clutch to the environment.

On the surface $r = R$, we formulate the thermal contact conditions as follows:

$$\left.\left[\frac{\partial T}{\partial r}\right]\right|_{r=R} = 0, \qquad \left.\left(\frac{\xi}{\zeta}\frac{\partial T^+}{\partial r} + [T]\right)\right|_{r=R} = 0. \tag{1.3}$$

The square brackets here denote the discontinuity of a quantity on the surface $r = R$ so that $[T] = T^+ - T^-$, where T^+ is the temperature of the shaft material for $r = R$, whereas T^- is the temperature of the clutch, ξ is the heat conduction coefficient, and ζ is the coefficient of the heat transfer from the clutch to the shaft. Because of the significant nonstationarity of the process, if the temperature difference of the contacting bodies is large ($T_* \gg T_0$) then we keep in the second equality in (1.3) the summand with the temperature gradient which is usually not taken into account.

Solving the temperature problem (1.1)–(1.3) does not pose significant difficulties. It suffices to use the popular software packages or create some program on the basis of finite-difference methods. Hence, the temperature distribution $T(r,t)$ for the assembly components can be considered as available (calculated) at each time after the moment of connection. Thus, the fit problem turns out to be a problem of calculating the thermal stresses caused by the nonstationary distribution of the temperature.

We assume that the material of the assembly components is isotropic and elastoplastic. The arising deformations d_{ij} are supposed to be small and combined of the invertible (elastic) e_{ij} and noninvertible (plastic) p_{ij} deformations:

$$d_{ij} = 0.5(u_{i,j} + u_{j,i}) = e_{ij} + p_{ij}. \qquad i,j = 1,2,3. \tag{1.4}$$

Here u_i are the components of the displacement vector in the rectangular coordinate system used below; the index after comma denotes differentiation with respect to the corresponding spatial coordinate. Invertible deformations and temperature define the stresses σ_{ij} in the deformed material [8]:

$$\sigma_{ij} = (\lambda e_{kk} - 3\alpha K\theta)\delta_{ij} + 2\mu e_{ij}, \qquad \theta = T - T_0. \tag{1.5}$$

Here λ, μ, and K are the elastic constants of the material (λ and μ are the Lame parameters, while $K = \lambda + 2\mu/3$ is the compression modulus).

When the stresses in the deformed material reach the loading surface $f(\sigma_{ij}) = 0$, its plastic flow is initiated. Assuming the conditions of the Mises maximum principle [9], the equation of this surface in the space of stresses plays the role of a plastic potential, and we have the associated law of plastic flow:

$$\varepsilon_{ij}^p = \frac{dp_{ij}}{dt} = \psi \frac{\partial f}{\partial t}, \qquad \psi > 0. \tag{1.6}$$

As the specific loading surface, we use the Tresca prism (the condition of maximum tangential stress) [9]:

$$\max |\sigma_i - \sigma_j| = 2k, \tag{1.7}$$

where σ_i are the principal values of the stress tensor and k is the yield strength. We assume that the latter is expressed as a given function of temperature. Since there is no such generally recognized dependence, we choose the simplest linear form:

$$k = k_0 T_m^{-1}(T_m - T). \tag{1.8}$$

Here k_0 is the yield strength of the assembly material at the room temperature and T_m is its melting temperature.

It is obvious that, for the above problem both suppositions are nonessential: taking for the yield strength the simplest linear dependence (1.8) on the temperature, as well as the assumption that the materials of the assembly components are the same. The calculations below can be repeated for the corresponding generalizations.

2. THERMOELASTIC DEFORMATION

Let us connect the beginning of the fit process with time $t = 0$. Following (1.4) and (1.5), we obtain the expressions for the stress components in the cylindrical coordinate system at time after $t = 0$:

$$\begin{aligned} \sigma_r &= (\lambda + 2\mu)u_{,r} + \lambda r^{-1}u - 3\alpha K\theta, \\ \sigma_\varphi &= \lambda u_{,r} + (\lambda + 2\mu)r^{-1}u - 3\alpha K\theta. \end{aligned} \tag{2.1}$$

Here $\sigma_r = \sigma_{rr}$, $\sigma_\varphi = \sigma_{\varphi\varphi}$, and $u(r,t) = u_r$ is the only nonzero component of the displacement vector. For $\theta = \theta_*$ this component is constant which holds for the clutch material prior to the start of the fit process. Inserting (2.1) into the equilibrium equation

$$\sigma_{r,r} + r^{-1}(\sigma_r - \sigma_\varphi) = 0,$$

we obtain the differential equation

$$u_{,rr} + r^{-1}u_{,r} - r^{-2}u = 0. \tag{2.2}$$

Hence,

$$u = C_1 r/2 + C_2/r. \tag{2.3}$$

The constants C_1 and C_2 are determined by the zero stress conditions on the free boundary of the clutch: $\sigma_r(R_1, 0) = 0$ and $\sigma_r(r_0, 0) = 0$. The equality $r = r_0$ determines the equation of the inner cylindrical surface of the clutch under the room temperature (prior to heating). Following the boundary conditions on the lateral surfaces of the clutch, we find: $C_1 = \beta\theta_*$ and $C_2 = 0$, where $\beta = 3\alpha K(\lambda + \mu)^{-1}$. For the technologically important size r_0, we obtain the value $r_0 = R(1 + \beta\theta_*)$. Under such uniform heating $\theta = \theta_* = T_* - T_0$, no stresses will be in the clutch material.

At time after $t = 0$ (during the fit process), the equation (2.3), as it follows from (2.1) and (2.2), becomes inhomogeneous with the right-hand side equal to $b\theta_{,r}$, $b = 3\alpha K(\lambda + 2\mu)^{-1}$. The solution of this equation should be written separately for the material of the shaft and the clutch. In the first case, we have

$$u(r,t) = br^{-1}\int_0^r \rho\theta(\rho,t)\,d\rho + C_{11}(t)r + C_{21}(t)r^{-1}; \tag{2.4}$$

and it follows for the clutch material that

$$u(r,t) = br^{-1}\int\limits_R^r \rho\theta(\rho,t)\,d\rho + C_{12}(t)r + C_{22}(t)r^{-1}. \tag{2.5}$$

We find the unknown functions of time in (2.4) and (2.5), fulfilling the continuity condition for the displacement $u(r,t)$ on the contact surface $r = R$, the condition $\sigma_r(R_1, 0) = 0$ on the free surface of the clutch, and the condition $u(0,t) = 0$ in the center of the shaft. Owing to the last requirement, we assume that $C_{21}(t) \equiv 0$. The dependencies for other integration functions are not shown here due to their cumbersomeness. Given the displacement field (2.4) and (2.5) determined by the temperature field $\theta(r,t)$, the stress in the assembly components can be found by inserting (2.4) and (2.5) into (2.1):
for $0 \le r \le R$,

$$\begin{aligned} \sigma_r &= -2\mu br^{-2}\int\limits_0^r \rho\theta(\rho,t)\,d\rho + 2(\lambda+\mu)C_{11}, \\ \sigma_\varphi &= 2\mu br^{-2}\int\limits_0^r \rho\theta(\rho,t)\,d\rho + 2(\lambda+\mu)C_{11}(t) - 2\mu b\theta(r,t); \end{aligned} \tag{2.6}$$

for $R \le r \le R_1$,

$$\begin{aligned} \sigma_r &= -2\mu br^{-2}\int\limits_R^r \rho\theta(\rho,t)\,d\rho + 2(\lambda+\mu)C_{12}(t) - 2\mu C_{22}(t)r^{-2}, \\ \sigma_\varphi &= 2\mu br^{-2}\int\limits_R^r \rho\theta(\rho,t)\,d\rho + 2(\lambda+\mu)C_{11}(t) + 2\mu C_{22}(t)r^{-2} - 2\mu b\theta(r,t). \end{aligned} \tag{2.7}$$

The above-obtained solution (2.4)–(2.7) is the initial condition for the origination and development of the subsequent process of a plastic flow. The moment of origination of the flow is connected with the fulfillment of the plasticity condition (1.7) which is reduced, in the considered case, to the requirement $|\sigma_r - \sigma_\varphi| = 2k$. Inserting (2.6) and (2.7) into this equality and taking into account (1.8), we see that the plastic flow arises in the clutch material at the place of its contact with the shaft (i.e., for $r = R$) at a certain time $t_0 > 0$. Beginning from this time, the elastoplastic boundary $r = m(t)$, $m(t_0) = R$ moves along the material of the clutch. Now the equilibrium equations should be integrated separately in the three domains: in the domain of the thermoelastic deformation of the shaft $0 \le r \le R$, of the similar deformation of the clutch $m(t) \le r \le R_1$, and in the domain of the flow of the clutch material $R \le r \le m(t)$. Furthermore, $m(t)$ remains unknown and to be determined while solving the problem. Moreover, the solution obtained under the listed conditions may be also bounded in time because the plasticity condition may be fulfilled for the shaft material, and then the new elastoplastic boundary will move from its surface $r = R$ towards the center of the shaft axis. However, the calculations showed that for realization of this effect it is necessary that $R_1 - R \ge R$. Because of this technical reason (and this can be rarely observed in practice), the given case, being of possible mathematical interest, is not considered below.

Despite the presence of the plastic flow domain $R \le r \le m(t)$, the formulas (2.4) and (2.6) remain valid. However, the functions of time arising in the integration with respect to the spatial coordinate need to be recalculated already on the basis of new conditions with the presence of the travelling boundary $r = m(t)$. Together with the unknown functions $C_{12}(t)$ and $C_{22}(t)$ for which new boundary conditions should be specified, the lower limit in the integral changes from R to $m(t)$. Furthermore, $m(t)$ remains unknown either. Proceeding from (1.5), we can write for the stress components in the flow domain:

$$\begin{aligned} \sigma_r &= (\lambda+2\mu)(u_{,r} - p_{rr}) + \lambda(r^{-1}u - p_{\varphi\varphi}) - 3\alpha K\theta(r,t), \\ \sigma_\varphi &= \lambda(u_{,r} - p_{rr}) + (\lambda+2\mu)(r^{-1}u - p_{\varphi\varphi}) - 3\alpha K\theta(r,t). \end{aligned} \tag{2.8}$$

In order to eliminate from (2.8) the components of irreversible deformations [10], it suffices to use the flow condition ($\sigma_r - \sigma_\varphi = -2k$) and the condition of irreversible incompressibility ($p_{rr} + p_{\varphi\varphi} = 0$). After the subsequent insertion of (2.8) into the equilibrium equation, we obtain the differential equation for the displacement $u(r,t)$:

$$u_{,rr} + (r^{-1}u)_{,r} - 2(\lambda+\mu)^{-1}(k_{,r} + r^{-1}k) = \beta\theta(r,t). \tag{2.9}$$

Integrating (2.9), we obtain the solution of the problem in the plastic flow domain, as before, up to unknown functions of time $C_{13}(t)$ and $C_{23}(t)$ arising as the integration constants (independent of r):

$$u(r,t) = 4\mu(\lambda+\mu)^{-1}r\int_R^r \rho^{-1}k(\rho,t)\,d\rho + \beta r^{-1}\int_R^r \rho\theta(\rho,t)\,d\rho + C_{13}r + C_{23}r^{-1},$$

$$\sigma_r = 2\int_R^r \rho^{-1}k(\rho,t)\,d\rho + 2(\lambda+\mu)C_{13}(t), \tag{2.10}$$

$$\sigma_\varphi = \sigma_r + 2k(\rho,t).$$

The irreversible deformations accumulating in the process of the flow are determined now according to the dependence

$$p_{rr}(r,t) = 0.5\beta\theta(r,t) - \beta r^{-2}\int_R^r \rho\theta(\rho,t)\,d\rho + 0.5(\lambda+2\mu)\mu^{-1}(\lambda+\mu)^{-1}k(r,t) + C_{23}(t)r^{-2}. \tag{2.11}$$

The unknown functions of time $C_{11}(t)$, $C_{21}(t)$, $C_{12}(t)$, $C_{22}(t)$, $C_{13}(t)$, $C_{23}(t)$, and $m(t)$ should be again determined from the boundary conditions and the conditions on the elastoplastic boundary. Their values are not given here either because of their bulkiness.

3.UNLOADING AND COOLING DOWN

In the process of equalizing of the temperatures of the connection elements, the conditions are created for slowing down the process of plastic flow and unloading. We have some time $t = t_1 > t_0$, beginning from which a new elastoplastic boundary $r = h(t)$ separates from the surface $r = R$ and then moves through the clutch material. Furthermore, in the domain $R \le r \le h(t)$, $h(t_1) = R$, the material again deforms reversibly, but there are already some accumulated irreversible deformations. If, as before, we calculate the stresses σ_r and σ_φ starting by the Duhamel–Neumann law (1.5) in the form (2.8) then, unlike (2.9), we cannot eliminate the irreversible deformations p_{rr} from the equilibrium equation with respect to the displacements in the domain $R \le r \le h(t)$. In the domain this equation has the form

$$u_{,rr} + (r^{-1}u)_{,r} - 2\mu(\lambda+2\mu)^{-1}\big(p_{rr}^{(r)} + 2p_{rr,r}^{(r)}\big) = b\theta_{,r}(r,t). \tag{3.1}$$

It should be noted that in (3.1) the irreversible deformations are the functions of just the spatial variable r and do not depend on time as in (2.11). Indeed, after passage of the elastoplastic boundary $r = h(t)$ through some material surface r of the clutch, the plastic deformations do not change anymore. The variables r and t are connected by this in (2.11), whereas the plastic deformations acquire a time-independent distribution in the domain $R \le r \le h(t)$. In the domains of thermoelastic deformation

$$0 \le r \le R \quad \text{and} \quad m(t) \le r \le R_1,$$

the relations (2.4)–(2.7) remain valid, which solve the problem. However, the unknown functions of time should be now searched again from the new conditions generated by the movement of the elastopalstic boundary surface $r = h(t)$. In the flow domain, the dependences (2.10) and (2.11) hold with the only difference that the lower integration limit should be inserted from R to $h(t)$. In the unloading domain $R \le r \le h(t)$, by (3.1), the relations hold:

$$u(r,t) = br^{-1}\int_R^r \rho\theta(\rho,t)\,d\rho + C_{14}(t)r + C_{24}(t)r^{-1} + 2\mu r\int_R^r \rho^{-1}p_{rr}(\rho)\,d\rho,$$

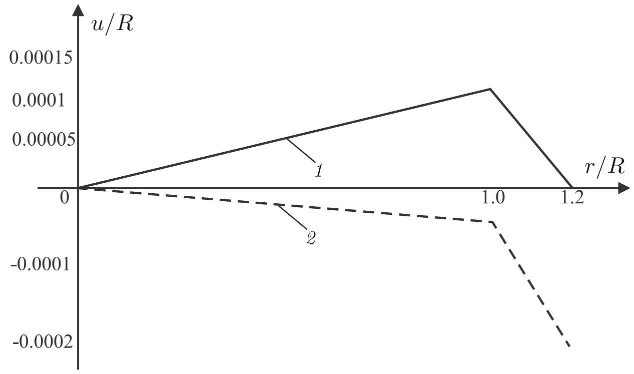

Fig. 1. Displacements in the assembly material before (*1*) and after (*2*) unloading

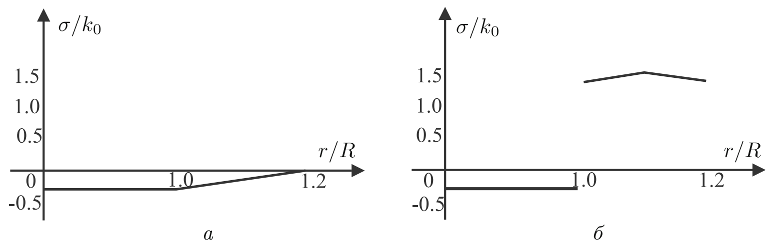

Fig. 2. Residual stresses σ_r (a) and σ_φ (b)

$$\sigma_r(r,t) = -2\mu b r^{-2}\int\limits_R^r \rho\theta(\rho,t)\,d\rho + 2(\lambda+\mu)C_{14}(t) - 2\mu r^{-2}C_{24}(t)$$
$$+ 4\mu(\lambda+\mu)(\lambda+2\mu)^{-1}\int\limits_R^r \rho^{-1}p_{rr}(\rho)\,d\rho, \qquad (3.2)$$

$$\sigma_\varphi(r,t) = 2\mu b r^{-2}\int\limits_R^r \rho\theta(\rho,t)\,d\rho + 2(\lambda+\mu)C_{14}(t) + 2\mu r^{-2}C_{24}(t)$$
$$+ 4\mu(\lambda+\mu)(\lambda+2\mu)^{-1}\int\limits_R^r \rho^{-1}p_{rr}(\rho)\,d\rho - 2\mu b\theta(r,t) + 4\mu(\lambda+\mu)(\lambda+2\mu)^{-1}p(r).$$

The functions $C_{14}(t)$ and $C_{24}(t)$ as all of the introduced earlier should again be determined together with $m(t)$ and $h(t)$ from the boundary conditions.

Over time, the surfaces $r = h(t)$ and $r = m(t)$ coincide, and the plastic flow domain ceases to exist. Denote this time by t_2 ($h(t_2) = m(t_2) = R_0$). The domain of the clutch material that has experienced irreversible deformations is thus contained within the limits $R \le r \le R_0$. Now the solution (2.10) and (2.11) is eliminated from consideration, but the integration function and the constant R_0 again has to be recalculated. Then the reversible (thermoelastic) deformation continues together with equalizing the temperatures of the assembly elements and the subsequent cooling of the constructions down to the room temperature. Note that the plastic flow takes place in the assembly material when the displacements have positive values (Fig. 1).

The conditions of tightness provide the residual stresses (Fig. 2) which turn out to be one and a half times lower if we take into account the plastic flow. This computational fact should be taken into account while assigning the technology parameters.

The main difficulty in algorithmization of the calculations by the above scheme is not only the use of the temperature distribution over the deformed body in the form of a numerical data array for their further application in the numerical-analytical procedures, but, also most importantly, tracking the distribution of the moving elastoplastic boundaries. On the opposite sides of these boundaries, the different systems of equations are solved; i.e., we have to fulfil the boundary conditions on the boundaries whose location is itself the solution of the problem. Even in the simplest case under consideration, this turned out to be far not a simple problem. In our disposal we have a specially developed computer code that allows us to carry the calculations including the cases of a more complicated geometry.

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