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Research article

Superparamagnetism in systems with magnetostatic interaction

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ABSTRACT

Superparamagnetism in systems with magnetostatic interaction is evaluated. Equations describing the dynamics of particle flipping under the influence of a magnetic field are presented, formulas for the relaxation time and magnetic susceptibility of particles are derived, and the frequency factor is estimated. Experimental data demonstrating a decrease in the susceptibility peak with increasing field are also discussed.

1. Introduction

In recent decades, there has been significant interest in nanomaterials due to their unique physical and chemical properties, making them suitable for a wide range of applications. Superparamagnetic nanoparticles are among the most intensively studied nanomaterials. These particles exhibit unique properties because their anisotropy energy can be comparable to the energy of thermal fluctuations, allowing for the reversal of their magnetic moment in a period defined by the Neel-Brown formula. Consequently, their magnetization dynamics are significantly dependent on the temperature regime and their particle size.

Relaxation phenomena in superparamagnetic particles, associated with processes that occur when the external magnetic field or temperature changes, are under active study. These processes impact the particles' response time and magnetic characteristics, which are significant for their practical applications. Early theoretical descriptions of magnetic relaxation relied on Langevin approximations, as well as the works of Kramers and Kolmogorov. A significant contribution to the development of the theory was made by Shliomis, whose work laid the foundations for understanding the dynamics of magnetization and relaxation in systems of superparamagnetic particles. In his seminal work, "Magnetic Fluids", Shliomis [1] suggested that the relaxation of magnetization in superparamagnetic systems is governed by two independent mechanisms (Néel relaxation and Brownian relaxation); his hypothesis enabled the explanation of the dynamics of magnetization in alternating fields and became the basis for interpreting experiments on the magnetic response of colloids. From the Fokker–Planck equation, he derived a macroscopic equation of motion for the magnetic moment of a suspension. Relaxation times were found to be proportional to the viscosity of the liquid and decreasing functions of the Langevin argument. Shliomis's work also demonstrated that particle anisotropy critically affects relaxation processes [2].

Kalmykov and Coffey made a significant contribution to the development of the theory of magnetic relaxation (see the monograph "Relaxation Phenomena in Condensed Matter Physics" [3]). They developed an improved model for the relaxation time of superparamagnetic particles, taking into account the influence of external magnetic fields, anisotropy, and damping. This model allows for a description of the dynamics of magnetization over a wide range of conditions, accounting for the influence of dipole–dipole interactions in particle ensembles, which is particularly important for dense colloids and magnetic fluids. They proposed analytical solutions for the Langevin and Fokker–Planck equations, applicable to systems with strong anisotropy [4,5].

Poperechny and Raikher [6] developed a theory of ferromagnetic resonance (FMR) for superparamagnetic nanoparticles with arbitrary nonaxial anisotropy. Their approach enabled the interpretation of experimental data. They demonstrated that multicomponent spectra, characteristic of highly anisotropic particles in the athermal limit, are preserved in superparamagnetic particles up to a certain temperature. They also found that the orientational distribution of the easy magnetization axes is not a necessary condition for the appearance of multi-peak absorption spectra in superparamagnetic systems. In 2023, Poperechny [7] investigated the linear response of a superparamagnetic nanoparticle suspended in a liquid in the presence of a stationary bias field. He confirmed that at zero bias field, the frequency dependence of the imaginary component of the dynamic susceptibility (absorption spectrum) features two maxima if the anisotropy energy exceeds the thermal energy several times. The presence of these peaks results directly from magnetic nanoparticles' bistability. He also showed that, in the presence of a bias field, the spectrum can acquire a third

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maximum if the Brownian rotation of the suspended particle slows down compared to the establishment of internal magnetic equilibrium within it.

The works of Eberbeck and Ludwig, who considered stochastic particle dynamics while accounting for non-Gaussian fluctuations and colored noise, should also be acknowledged. In particular, they applied the generalized Fokker–Planck equations to systems with multiple relaxation time scales. These approaches proved to be especially significant when analyzing nonlinear magnetic susceptibility, which is crucial to the examination of EPR and NMR signals [8,9].

Numerical methods, particularly the Monte Carlo and Langevin dynamics methods for modeling interacting ensembles, have made significant progress. The studies by P. Ilg [10–13] and D. Berkov [14–16] demonstrate the possibility of quantitatively describing magnetization dynamics, including hysteresis and nonlinear susceptibility. The approach used in [17] is based on the Fokker–Planck–Brown equation. The authors incorporate interparticle interactions and magnetic anisotropy to model the dynamic response of superparamagnetic nanoparticles in an alternating field, illustrating how these factors affect susceptibility and relaxation time. Kuntscher et al. [18] utilize micromagnetic modeling, which helps in studying the evolution of relaxation time in the presence of interparticle interactions, which is crucial for understanding the hyperthermia of magnetic fluids. Of particular interest is the study of the blocking temperature of superparamagnetic particles, defined as the temperature below which the particles exhibit collective magnetic behavior with hysteresis. Monte Carlo simulations of these processes allow for detailed investigations of the magnetic properties of particles and comparisons of the results with experimental data, contributing to a deeper understanding of the physical mechanisms underlying superparamagnetism. In particular, Gogoi et al. [19] investigated the blocking temperature in the context of new synthetic approaches for creating superparamagnetic particles. However, all these approaches require significant computational resources and are generally limited by the number of particles and time scales.

Recent studies indicate that superparamagnetic particles can be used for targeted drug delivery, particularly in oncology, as they can deliver microRNA and other therapeutic agents directly to tumor cells. For instance, the effects of highly hydrophilic superparamagnetic particles on macrophage function have been investigated, which is crucial for understanding their interactions with biological systems [20,21].

Thus, the study of superparamagnetic particles remains a significant area of modern science, presenting numerous opportunities for further fundamental and applied research.

The simplest classical method for describing the properties of magnetic systems with exchange interaction is the molecular field theory. However, it cannot fully account for the entire variety of magnetic ordering types. One approach that expands application possibilities while retaining the molecular field theory's simplicity is the method of random fields of exchange interaction, which has been developed by many authors. The general approach to determining the distribution density of random fields was formulated by Chandrasekhar and was successfully utilized to study the magnetic properties of interacting particle systems in the works of Anderson, Berkov, Meshkov, and Shcherbakov [22–24]. In 1992, Belokon [25] proposed applying the method of random fields of exchange interaction for dilute magnets with direct exchange. In 2002, Belokon and Nefedev [26] determined an approximate distribution density of random fields in the form of a normal distribution. In this case, the mean value and variance were defined by the law of particle interaction. This could involve either direct exchange or indirect interaction. The advantage of this approach over conventional molecular field theory is that it allows for a quantitative description of phase transitions in systems with any exchange law, takes into account the possibility of diffusion, and enables estimation of the critical concentration of interacting particles below which a phase transition is impossible [27–31].

In this paper, the method is used to describe the behavior of the magnetic susceptibility of a system of superparamagnetic particles with magnetostatic interaction at low temperatures. Most previous works (e.g., Shliomis, Kalmykov, Coffey, and Poperechny) focused either on relaxation processes considering the dipole–dipole interaction or on solutions of the Langevin and Fokker–Planck equations. However, they did not account for the contribution of the exchange interaction through a stochastic approach, especially with a variable distribution of anisotropy and size parameters. The proposed model considers the heterogeneity of the nanoparticle ensemble, particularly the size distribution and anisotropic barriers, which aligns the theory more closely with real experimental systems.

The article demonstrates that fluctuations in the exchange interaction between particles significantly affect the position of the magnetic susceptibility peak (blocking temperature) and the shape of the temperature dependence of magnetization. An analytical and numerically stable description of the temperature evolution of the magnetization in an ensemble of superparamagnetic particles is proposed, factoring in the intensity of exchange interactions and their fluctuations. This approach broadens the applicability of classical models of superparamagnetism, such as the Brown, Langevin, or Kalmykov models, to more complex systems exhibiting collective effects. The article is the first to describe, within the random fields' method of exchange interaction, the transition from independent to collective behavior in ensembles of superparamagnetic particles with variable intensity of exchange interaction, both qualitatively and quantitatively.

2. The method of random interaction fields

The method of random interaction fields is a method of molecular (effective) field, which is considered as a random variable [26]. The Heisenberg Hamiltonian of a system of interacting particles (ions) has the form:

$$\mathcal{H} = -\sum_{i<k} J_{ik} S_i S_k - g\mu_B H_l \sum_i S_i, \tag{1}$$

where S_i are the components of the spin vector, g is the Lande multiplier, H is the external magnetic field, and μ_B is the Bohr magneton. Since $g\mu_B S_i$ is the magnetic moment of the ion m, the classical Hamiltonian can be written as follows:

$$\mathcal{H} = -\sum_{i<k} J_{ik} m_i m_k - H_l \sum_i m_i. \tag{2}$$

The summation in the first term is over all pairs of particles. In the Ising model, the magnetic moment can have only two possible orientations: $+m$ (up), and $-m$ (down). If we assume the possibility of replacing exchange interacting particles at some lattice sites with "frozen-in" "non-magnetic" impurities with a density of $1 - p$, we obtain a model with dilution across sites. The term $m_i J_{ik} m_k$ can be considered as the energy of the magnetic moment m_i in the exchange interaction field $H_{ik} = J_{ik} m_k$. In the molecular field theory (exchange interaction field), a separate ion is considered $m_i \equiv m$ interacting with the environment. If $H = 0$, then

$$\mathcal{H} = -\sum_i m_i \sum_k J_{ik} m_k = -\sum_i m_i \sum_k H_{ik} = -\sum_i m_i H_i, \tag{3}$$

where $H_i = \sum_k J_{ik} m_k$ is the total field of exchange interaction on m_i, $\varphi_k = H_{ik}$ the index i omitted.

The dependence of the exchange integral on the distance determines the number of terms to consider in the sum. This sum is a random variable, the distribution density of which, in some approximation, as shown in our studies [25,26], has the form:

$$W(H, M) = \frac{1}{\sqrt{\pi} B} \exp\left\{ -\frac{\left[H - H_0 M\right]^2}{B^2} \right\}, \tag{4}$$

where the average value of $\langle H \rangle = H_0 M$ and dispersion of $2\sigma^2 = B^2$ are expressed in terms of the p concentration of exchange-interacting

particles, the φ_k effective field of exchange interaction produced by an atom with number k, and configurationally and thermodynamically averaged magnetic moment M as follows:

$$H_0 = p\sum_k \varphi_k, B^2 = 2p\left[1 - M^2 p\right]\sum_k \varphi_k^2. \tag{5}$$

The equation determining the dependence of relative average magnetic moment M on temperature and the atomic concentration has the form:

$$M = \int \tanh\left[\frac{mH}{kT}\right] W(H, M)\mathrm{d}H, \tag{6}$$

where m is the magnetic moment of the ion. $W(H, M)$ is a "smeared" δ - function.

Then, Eq. (6) will be as follows:

$$M = \frac{1}{\sqrt{\pi}B}\int \tanh\left(\frac{m\left(H + H_0 M\right)}{kT}\right)\exp\left(-\frac{H^2}{B^2}\right)\mathrm{d}H. \tag{7}$$

Simple estimates can be obtained by replacing the Gaussian distribution function with approximate function $f(H)$:

$$f(H) = \begin{cases} 0, & H > B,\ H < -B \\ \frac{1}{2B}, & -B < H < B. \end{cases} \tag{8}$$

Examples of the numerical solution of Eq. (7) with exact and approximate functions, from which it follows that near the phase transition points, where the M values are small, the error in the calculations is insignificant, are given in [26].

For small M values, in this case, we have that

$$M = \frac{1}{2B}\int_{-B}^{B} \tanh\left(\frac{m\left(H + H_0 M\right)}{kT}\right)\mathrm{d}H. \tag{9}$$

3. Preisach-Néel diagrams in the study of superparamagnetism

The Preisach–Néel diagram is an important tool for analyzing hysteresis phenomena in magnetic materials, including superparamagnetic particles. The diagram model was developed to describe the behavior of magnetic domains in ferromagnets but has been adapted to study superparamagnetism and other magnetic systems, demonstrating the versatility and power of the diagram in understanding complex magnetic phenomena. The Preisach–Néel diagram is based on the concept that a system of magnetic particles is considered a set of elementary hysteresis units known as hysterons. Each hysteron is a miniature magnetic domain that can switch between two stable states under the influence of an external magnetic field. These states have specific coercivities and magnetic field variations, which allows the modeling of the overall magnetic response of the system as the sum of the individual reactions of all hysterons.

In superparamagnetic systems, blocking temperature plays a key role because below this temperature, particles start to exhibit collective magnetic behavior with hysteresis. At the same time, above the temperature, the particles behave as independent dipoles and hysteresis disappears. The Preisach–Néel diagram allows modeling of this transition and an analysis of the changes in hysteresis properties at different temperatures. The Preisach model is also used for detailed modeling of the magnetic properties of superparamagnetic particles, including the effect of the frequency of magnetic field change and the interactions between particles. This modeling is especially important for optimizing the magnetic properties of nanoparticles used in biomedical technologies.

Let us consider, in the framework of the Ising model, a system of ferromagnetic particles of the same volume V distributed over critical fields of magnetization reversal H_c in the interval $0 \le H_c \le D$ ($H_c = KI_s$, where K is the constant of shape anisotropy, I_s is spontaneous magnetization, and D is the maximum possible critical field). The axes of the easiest magnetization are oriented in one direction. Further, the external field acts in the same direction. Moreover, the hysteresis loop

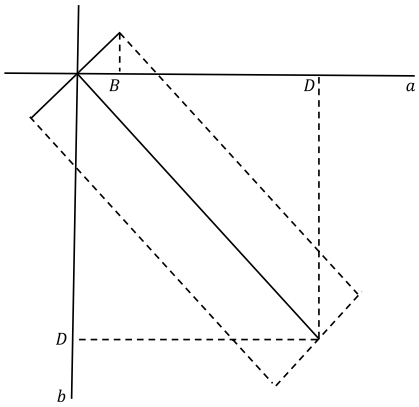

Fig. 1. Preisach– Néel diagram for magnetic moments of particles in a state of equilibrium. The coordinates of the particle: $a = H_c + H_{int}$ and $b = H_c - H_{int}$, where H_c is the critical field of magnetization, H_{int} is the field of magnetostatic interaction, D is the maximum possible critical field, B is the maximum field of magnetostatic interaction.

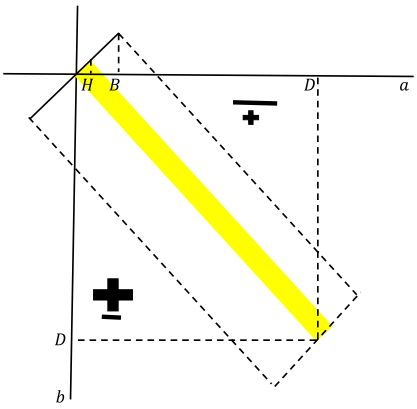

Fig. 2. Preisach–Néel diagram for magnetic moments of particles in a state of nonequilibrium. In the upper region above the equilibrium line, particles with a magnetic moment directed downwards predominate; in the lower region, particles with a magnetic moment directed upwards predominate. H is the external applied field that shifts the equilibrium line, D is the maximum possible critical field, B is the maximum field of magnetostatic interaction.

of the particles is rectangular. The field of magnetostatic interaction H_{int} is contained in the interval $-B \le H_{int} \le B$, where B is of the order $p \cdot I_s$. The relaxation time for such particles is comparable to experimental time. It is convenient to depict such a system on the Preisach–Néel diagram (Fig. 1), where the fields $a = H_c + H_{int}$ and $b = H_c - H_{int}$ act as the coordinates of the particle.

In a state of equilibrium on the line $a = b$ the number of magnetic moments of particles oriented "up" N_+ and "down" N_- is the same; but as we move away from this line, the ratio N_+ to N_- changes. Furthermore, due to symmetry, the total magnetic moment will be equal to 0. Turning on a small field $H \ll B$ will lead to a shift in the line of equality N_+ and N_-, and a region responsible for the excess N_+will appear on the diagram N_-. This is a region of uncompensated magnetic moments with a width $2H$ and a length D (Fig. 2).

Therefore, the initial conditions are

$$N_+ + N_- = N_0, \tag{10}$$

where N_0 is the total number of particles. Over a period t, the situation will change to

$$\frac{dN_+}{dt} = -\lambda_{21}N_+ + \lambda_{12}N_-, \tag{11}$$

$$\frac{dN_-}{dt} = -\lambda_{12}N_- + \lambda_{21}N_+, \tag{12}$$

$$\frac{dN_0}{dt} = 0, \tag{13}$$

where λ_{12} is the probability of the particle's magnetic moment flipping from "bottom" to "up", λ_{21} is the probability of the magnetic moment of a particle flipping from "top" to "bottom". The solution to systems (11) and (12) has the form

$$N_+ = A\exp\left[-\left(\lambda_{12}+\lambda_{21}\right)t\right] + \frac{\lambda_{12}}{\left(\lambda_{12}+\lambda_{21}\right)}N_0, \tag{14}$$

where $\frac{1}{\tau} = \lambda_{12} + \lambda_{21}$ and τ is the relaxation time.

When $t \gg \tau$, the first term becomes 0 and only the expression below remains:

$$N_+ = \widetilde{N}_+ = \frac{\lambda_{12}}{\left(\lambda_{12}+\lambda_{21}\right)}N_0. \tag{15}$$

Accordingly, based on formula (10):

$$N_- = \frac{\lambda_{21}}{\left(\lambda_{12}+\lambda_{21}\right)}N_0. \tag{16}$$

In our model, the average magnetic moment of magnetostatically interacting particles is determined by the difference in the magnetic moments of particles oriented "up" and "down":

$$\widetilde{M} = \mu\left(N_+ - N_-\right) = \mu\frac{\lambda_{12}-\lambda_{21}}{\left(\lambda_{12}+\lambda_{21}\right)}N_0, \tag{17}$$

where μ is the magnetic moment of the particle $\mu = I_s V$.

4. Superparamagnetism in a system with magnetostatic interaction

As follows from experimental data [32], at sufficiently low temperatures, the magnetic moment increases proportionally to $\log t$, where t is the time of action of the constant field H. The ensemble of particles should have a wide spectrum of relaxation times, which is due to their spread over the volume and magnetization reversal fields H_c. The probability of fluctuation is determined by the minimum work R required to overcome the potential barrier:

$$\lambda = f_0\exp\left[-\frac{R}{kT}\right]. \tag{18}$$

Here, f_0 is the frequency factor, and k is the Boltzmann constant. In the absence of an external field, the critical value of the magnetization reversal field is $\frac{H_c}{2}$. The external field H and the interaction field H_{int} either help or hinder the reversal of the magnetic moment, contributing to the work R. When the magnetostatic interaction field H_{int} and the field H are oriented "up", they help the particle overcome the barrier $\frac{H_c}{2}$, but hinder its reversal "down". In this case, the relaxation time τ, the magnetic moment of the particles, and its critical field will be related by the relation:

$$\frac{1}{\tau} = \lambda_{12} + \lambda_{21}, \tag{19}$$

$$\lambda_{12} = f_0\exp\left[-\frac{\mu}{kT}\left(\frac{H_c}{2} - \left(H_{int}+H\right)\right)\right], \tag{20}$$

$$\lambda_{21} = f_0\exp\left[-\frac{\mu}{kT}\left(\frac{H_c}{2} + \left(H_{int}+H\right)\right)\right]. \tag{21}$$

From expressions (19)–(21), it follows that

$$\frac{1}{f_0\tau} = \exp\left[-\frac{\mu H_c}{2kT}\right]2\cosh\left[\frac{\mu\left(H_{int}+H\right)}{kT}\right]. \tag{22}$$

Critical fields of particles for which equilibrium has been established (precisely, $t = \tau$ and t is the time of action of field H) are:

$$H_c = \frac{2kT}{\mu}\left(\ln\left[f_0 t\right] + \ln\left[2\cosh\left[\frac{\mu\left(H_{int}+H\right)}{kT}\right]\right]\right). \tag{23}$$

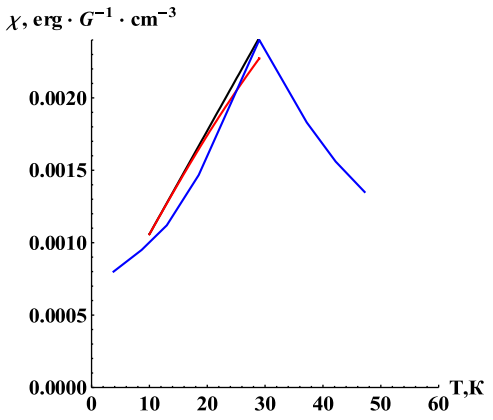

Fig. 3. Graphs of the dependence of the magnetic susceptibility of the $\chi(T)$ AuFe alloy with an iron concentration of 8% at field values $H = 10$ Oe (black line), $H = 100$ Oe (red line), experimental data [33] (blue line).

Expression (23) allows us to find the transition temperature of all particles to the superparamagnetism state depending on H_{int} and H, when the maximum field $H_c = D$. As a well-known example of the behavior of such particles, we consider an AuFe alloy with an iron concentration of 8%; the experimental data and graphs of the initial magnetic susceptibility's dependence on temperature are described in [33]. At low temperatures, the behavior of this alloy's magnetic moment and spontaneous magnetization are independent of temperature. According to the theory of random interaction fields, at a temperature below the critical temperature, the leaky cluster is destroyed, and a system of finite-sized clusters consisting of iron ions is formed. The spontaneous magnetization of iron $I_s = 1700\ erg\cdot G^{-1}\cdot\ \mathrm{cm}^{-3}$ [34]. The relaxation time for such clusters will be determined by the shape and number of particles. The average field of magnetostatic interaction is on the order of $pI_s = 0.08\cdot 1700 = B = 1360\ Oe$. For an iron particle, the average magnetization reversal field is on the order of 1700 Oe since the shape anisotropy constant is on the order of unity ($H_c = KI_s = 1\cdot 1700 = 1700\ Oe$). However, for small particles, for which the number of surface ions is relatively large, the critical magnetization reversal field is significantly smaller [35]. The experimental point of maximum magnetic susceptibility given in [33] was taken as a guideline when choosing the critical field $D = 800\ Oe$. The magnetic moment μ was found from the relation $\frac{2kT}{\mu}\ln\left[t\cdot 10^{13}\cdot 2\cdot\cosh\left[\frac{B\mu}{kT}\right]\right] = D + B$ because, at this point, all the particles have passed into the superparamagnetic state. The Boltzmann constant $k = 1.38\cdot 10^{-16}\mathrm{erg}\cdot\mathrm{K}^{-1}$. There are various opinions in the literature regarding the frequency factor; in this paper, we associate it with the Debye temperature, which, for example, for gold or iron is of the order of 10^2 K. The probability of an irreversible rotation of the magnetic moment of a cluster is determined by the "frequency of fluctuation attempts", which is associated with elastic oscillations of ions in the crystal lattice. By definition, the Debye frequency is the maximum possible frequency of the phonon spectrum. Therefore, we associate the frequency factor with the Debye frequency. Accordingly, from the expression, $\hbar\omega_D = kT_D$, the estimate $\omega_D \sim f_0$ is of the order of $10^{13}s^{-1}$. If we set $H_{int} = 0$ and $H = 0$, then, based on formula (23), the temperature of the transition of particles to the superparamagnetic state is $T \approx 30$ K, which corresponds to the experimental value obtained in the literature [33] and presented in (Fig. 3).

The average magnetic moment of magnetostatically interacting particles $\widetilde{M}$, per particle for a given H_{int} can be found from formulas (17)–(21):

$$\widetilde{M} = \mu\frac{\lambda_{12}-\lambda_{21}}{\left(\lambda_{12}+\lambda_{21}\right)} = \mu\tanh\left[\frac{\mu\left(H_{int}+H\right)}{kT}\right]. \tag{24}$$

The number of particles in the Preisach–Néel diagram from fields $dH_c\ dH_{int}$ is determined from the relationship

$$dN = \frac{N_0}{2BD} dH_c\ dH_{int}. \tag{25}$$

In the framework of the random interaction field method, the total magnetic moment of a system of particles at time t is defined as

$$M_0 = \frac{\mu N_0}{2BD} \int_{-B}^{B} \frac{2kT}{\mu} \tanh\left[\frac{\mu\left(H_{int}+H\right)}{kT}\right] \ln\left[2f_0 t \cosh\left[\frac{\mu\left(H_{int}+H\right)}{kT}\right]\right] dH_{int}. \tag{26}$$

At a fixed time t, the dependence of susceptibility χ to temperature$T$ can be considered as $\partial M/\partial H$, taking into account that in this temperature range, $I_s(T) = I_s(0)$. Fig. 3 shows graphs of the dependence of the magnetic susceptibility of the $\chi(T)$ AuFe alloy with an iron concentration of 8% at field values $H = 10\ Oe$ (black line), $H = 100\ Oe$ (blue line). The experimental graph corresponds to the graph obtained as a result of the calculation at $H = 10\ Oe$. At temperatures of up to 30 K, particles acquire a magnetic moment in the direction of the applied field; and the process extends to the maximum value H_c. It is apparent that after reaching the temperature at which all particles pass into the superparamagnetic state, further change in susceptibility is reduced due to the randomization of the directions of the magnetic moments of the particles due to an increase in temperature. If the field is turned on, then an increase in the field will cause a fall in the peak of susceptibility, correlating with the experimental data for the AuFe alloy with an iron concentration of 2% (see Figure in [33]). For clarity of comparison with experimental data, it is convenient to choose the total number of particles $N_0 = 2 \cdot 10^{15}$.

As shown in formula (26), the time required to reach the superparamagnetic state $2f_0 \cosh\left[\frac{\mu\left(H_{int}+H\right)}{kT}\right] t$ depends on the fields H_{int} and H. Therefore, at $H > B$ this time decreases, which leads to reaching the maximum value of the magnetic moment at a low temperature.

5. Estimation of the magnitude of the frequency factor

The formulas obtained allow us to estimate the magnitude of the frequency factor. Let the magnetization of the particle system be obtained over a long period t_0 at a temperature T and with the field turned on:

$$H_c = \frac{2kT}{\mu} \ln\left[2f_0 t_0\right]. \tag{27}$$

Magnetization is destroyed by the temperature T during the same period. If heating is carried out instantaneously from temperature t_0 to $T + \Delta T$, then, from the formula, it will follow that

$$H_c = \frac{2k(T+\Delta T)}{\mu} \ln\left[2f_0 t_1\right]. \tag{28}$$

The time of destruction of magnetization t_1 will be short, that is, the destruction process will be accelerated. Solving Eqs. (27) and (28) together, we obtain the expression

$$\ln\left[\frac{t_0}{t_1}\right] = \frac{\Delta T}{T} \ln\left[2f_0 t_0\right]. \tag{29}$$

Formula (29) allows us to find the frequency factor f_0 for known values t_0, t_1, T, and ΔT.

Because the heating process takes a certain duration, the time interval t_0 for the formation of magnetization must be large enough so that the transition to the decay curve is completed during the process of magnetization destruction at $T + \Delta T$.

6. Conclusion

This study was conducted within the framework of a simple superparamagnetism model in systems with magnetostatic interaction. It has been established that:

1. The simple model evaluated allows one to accurately describe the behavior of the magnetic susceptibility of a system of superparamagnetic particles with magnetostatic interaction at low temperatures.
2. Magnetic susceptibility significantly depends on an external field when the former reaches maximum interaction fields.
3. The maximum value of susceptibility in large fields is observed at low temperatures.
4. There is a fundamental possibility of experimentally assessing the frequency factor when observing the destruction of residual magnetization obtained over a sufficiently long time.

The study relied on known experimental data obtained at low temperatures, allowing us to simplify some formulas.

Such calculations can be carried out for a system of particles exhibiting superparamagnetic properties at high temperatures. In this case, the temperature dependence of spontaneous magnetization, which determines critical fields and interaction fields, should be taken into account.

In conclusion, this study significantly deepens our understanding of superparamagnetism and its manifestations in nanoscale systems. Further research in this area will focus on studying the influence of various factors, such as particle shape and size, on the magnetic properties of nanoscale systems as well as developing new theoretical models to describe complex magnetic phenomena.

CRediT authorship contribution statement

V. Belokon: Writing – review & editing, Methodology, Investigation. **O. Dyachenko:** Writing – original draft, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] M.I. Shliomis, Magnetic fluids, Sov. Phys. Uspekhi 17 (2) (1974) 153.
[2] M. Martsenyuk, Y.L. Raikher, M. Shliomis, On the kinetics of magnetization of suspensions of ferromagnetic particles, Zh. Eksp. Teor. Fiz. 65 (1973) 834–841.
[3] W.T. Coffey, Relaxation Phenomena in Condensed Matter, vol. 186, John Wiley & Sons, 2009.
[4] Y.P. Kalmykov, Longitudinal dynamic susceptibility and relaxation time of superparamagnetic particles with cubic anisotropy: effect of a biasing magnetic field, Phys. Rev. B 61 (9) (2000) 6205.
[5] W.T. Coffey, Y.P. Kalmykov, Thermal fluctuations of magnetic nanoparticles: Fifty years after brown, J. Appl. Phys. 112 (12) (2012).
[6] I. Poperechny, Y.L. Raikher, Ferromagnetic resonance in uniaxial superparamagnetic particles, Phys. Rev. B 93 (1) (2016) 014441.
[7] I. Poperechny, Multipeak dynamic magnetic susceptibility of a superparamagnetic nanoparticle suspended in a fluid, Phys. Rev. B 107 (6) (2023) 064416.
[8] J. Dieckhoff, D. Eberbeck, M. Schilling, F. Ludwig, Magnetic-field dependence of Brownian and Néel relaxation times, J. Appl. Phys. 119 (4) (2016).
[9] N. Löwa, P. Knappe, F. Wiekhorst, D. Eberbeck, A.F. Thünemann, L. Trahms, Hydrodynamic and magnetic fractionation of superparamagnetic nanoparticles for magnetic particle imaging, J. Magn. Magn. Mater. 380 (2015) 266–270.
[10] P. Ilg, M. Kröger, S. Hess, A.Y. Zubarev, Dynamics of colloidal suspensions of ferromagnetic particles in plane couette flow: Comparison of approximate solutions with Brownian dynamics simulations, Phys. Rev. E 67 (6) (2003) 061401.
[11] P. Ilg, M. Kröger, Anisotropic self-diffusion in ferrofluids studied via Brownian dynamics simulations, Phys. Rev. E Stat. Non Linear Soft Matter Phys. 72 (3) (2005) 031504.

[12] P. Ilg, Diffusion-jump model for the combined Brownian and Néel relaxation dynamics of ferrofluids in the presence of external fields and flow, Phys. Rev. E 100 (2) (2019) 022608.
[13] P. Ilg, Equilibrium magnetization and magnetization relaxation of multicore magnetic nanoparticles, Phys. Rev. B 95 (21) (2017) 214427.
[14] A. Lyberatos, D.V. Berkov, R.W. Chantrell, A method for the numerical simulation of the thermal magnetization fluctuations in micromagnetics, J. Phys.: Condens. Matter. 5 (47) (1993) 8911.
[15] D.V. Berkov, Fast switching of magnetic nanoparticles: Simulation of thermal noise effects using the Langevin dynamics, IEEE Trans. Magn. 38 (5) (2002) 2489–2495.
[16] D. Berkov, N. Gorn, R. Schmitz, D. Stock, Langevin dynamic simulations of fast remagnetization processes in ferrofluids with internalmag- netic degrees of freedom, J. Phys.: Condens. Matter. 18 (38) (2006) S2595.
[17] A.V. Ambarov, V.S. Zverev, E.A. Elfimova, Numerical modeling of the magnetic response of interacting superparamagnetic particles to an ac field with arbitrary amplitude, Modelling Simul. Mater. Sci. Eng. 28 (8) (2020) 085009.
[18] A. Kuncser, N. Iacob, V.E. Kuncser, On the relaxation time of interacting superparamagnetic nanoparticles and implications for magnetic fluid hyperthermia, Beilstein J. Nanotechnol. 10 (1) (2019) 1280–1289.
[19] B. Gogoi, U. Das, Review on magnetism in nanomaterials and superparamagnetism, in: Selected Progresses in Modern Physics: Proceedings of TiMP 2021, Springer, 2021, pp. 303–312.
[20] G. Kara, B. Ozpolat, Spions: Superparamagnetic iron oxide-based nanoparticles for the delivery of micrornai-therapeutics in cancer, Biomed. Microdevices 26 (1) (2024) 16.
[21] E. Korakaki, Y.V. Simos, N. Karouta, K. Spyrou, P. Zygouri, D.P. Gournis, K.I. Tsamis, H. Stamatis, E. Dounousi, P. Vezyraki, et al., Ef- fect of highly hydrophilic superparamagnetic iron oxide nanoparticles on macrophage function and survival, J. Funct. Biomater. 14 (10) (2023) 514.
[22] S. Chandrasekhar, Stochastic problems in physics and astronomy, Rev. Modern Phys. 15 (1) (1943) 1.
[23] D. Berkov, S. Meshkov, Theory of remagnetization curves of dilute random magnets, J. Exp. Theor. Phys. 67 (1988) 2255.
[24] V. Shcherbakov, The distribution function of molecular fields in systems with randomly distributed interaction centres, Fiz. Met. Metalloved. 48 (6) (1979) 1134–1137.
[25] V. Belokon, S. Semkin, Random-field method in the king model of a diluted ferromagnet, Zh. Eksp. Teor. Fiz. 102 (1992) 1254–1258.
[26] V. Belokon', K. Nefedev, Magnetic phase transitions in amorphous sys- tems with competing exchange interactions, Phys. Solid State 44 (2002) 1708–1710.
[27] V. Belokon, V. Kapitan, O. Dyachenko, The combination of the random interaction fields' method and the Bethe–Peierls method for studying two-sublattice magnets, J. Magn. Magn. Mater. 401 (2016) 651–655.
[28] V. Belokon, A. Trofimov, O. Dyachenko, Oguchi's method and random interaction fields' method: Investigation of properties of ferromagnetic materials, J. Magn. Magn. Mater. 471 (2019) 501–503.
[29] V. Belokon, O. Dyachenko, Random interaction fields method: Magnetic phase transitions in the thin films, J. Magn. Magn. Mater. 374 (2015) 92–95.
[30] V. Belokon, R. Lapenkov, E. Chibiriak, O. Dyachenko, Magnetic sus-ceptibility of systems with different types of interactions: The random interaction fields method, J. Magn. Magn. Mater. 512 (2020) 167051.
[31] V. Belokon, R. Lapenkov, O. Dyachenko, Magnetic phase transition in an amorphous alloy: The theory of random fields of exchange interaction, J. Magn. Magn. Mater. 564 (2022) 170172.
[32] J.-L. Dormann, D. Fiorani, E. Tronc, Magnetic relaxation in fine-particle systems, Adv. Chem. Phys. 98 (1997) 283–494.
[33] V. Cannella, J.A. Mydosh, Magnetic ordering in gold-iron alloys, Phys. Rev. B 6 (11) (1972) 4220.
[34] H. Danan, A. Herr, A. Meyer, New determinations of the saturation magnetization of nickel and iron, J. Appl. Phys. 39 (2) (1968) 669–670.
[35] L. Afremov, Y.V. Kirienko, A. Petrov, A. Chepak, Size effect's influence on the magnetic phase transitions in the nanosized magnets, J. Supercond. Nov. Magn. 36 (2) (2023) 587–600.